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PLANE TRIGONOMETRY.

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AN ELEMENTARY TREATISE

ON

PLANE TRIGONOMETRY;

WITH

A Numerous Collection of Examples,

CHIEFLY DESIGNED FOR THE USE OF SCHOOLS AND BEGINNERS.

BY

R. D. BEASLEY, M.A.

FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE.

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BIBLIOTHECA · BODLEIANA ·

PREFACE.

THE following pages are specially intended for use in Schools. In the choice of matter I have been chiefly guided by the requirements of the three days' Examination at Cambridge, with the exception of proportional parts in logarithms, which I have omitted. In one point, I have ventured to deviate from the usual custom. I have denoted angles throughout by the Greek characters, even before the explanation of circular measure. My reasons for doing so were chiefly two; first, that they are much more distinctly and easily written, and it is advisable that a boy should be accustomed in the text to the notation he uses on his paper; secondly, I am thus enabled to insert the algebraical symbols for the angles in the figures, a very great advantage in such propositions as that proved in Art. 27. I have added about 400 examples mainly collected from the Examination Papers of the last ten years, and I have taken great pains to exclude from the body of the work any which might dishearten a beginner by their difficulty.

In conclusion, I must express my great obligations to my friend the Rev. R. B. Mayor, of Rugby. Without his kind encouragement I should not have ventured to have offered these pages to the public; and I am indebted to him, not only for many valuable suggestions, but for a careful revision of the whole work.

R. D. BEASLEY.

ST JOHN'S COLLEGE, CAMBRIDGE,
September, 1858.

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PLANE TRIGONOMETRY.

SECTION I.

UNITS OF MEASUREMENT. USE OF SIGNS + AND −. MEANING OF THE TERM "ANGLE" IN TRIGONOMETRY. THE TRIGONOMETRICAL RATIOS. PRACTICAL APPLICATION. INSTRUMENTS FOR SURVEYING.

1. *Object of Trigonometry.* In Trigonometry we apply Algebraical symbols to establish certain relations between the magnitudes of the sides and angles of plane rectilineal figures. These relations are useful for all the higher branches of Mathematics, and are specially applicable to surveying, and the determining the heights and distances of inaccessible objects. In the present treatise we shall confine ourselves to the simpler relations, and the practical application of them.

We must first consider the mode of estimating algebraically the magnitudes of lines, areas, and angles.

2. *Measurement of lines.* As lines have neither breadth nor thickness we have only to measure their length. To do this we take some standard length, as one foot, one inch, five inches, or any other definite length as our unit of measurement, and the length of any line is then measured and represented by the number, whether whole or fractional, of these units which it contains. Thus if 5 inches is our unit, a line of 20 inches is measured by the number 4, and is called the line 4, that is, the line whose length is 4 times the unit of length. So also the line a would be the line whose length is a times the unit of length.

In investigations merely involving algebraical symbols it is indifferent what unit is used, only we must be careful to remember that throughout the same investigation we use the same unit. Thus if two lines a, b enter into our calculations, we must consider the length of the line a to be a times *some* unit of length, and of the line b to be b times the *same* unit. When we apply our general results to numerical examples we must be careful to consider the special unit employed.

3. In the measurement of superficies we take the square of which one side is the unit of length, as the unit of superficies; and then any area is measured by the number of these units it contains.

N.B. The area of a rectangle whose sides contain a, b linear units respectively, contains ab superficial units. Also the areas of a parallelogram and a triangle whose bases are b and altitudes a are ab and $\frac{ab}{2}$ respectively.

4. *Measurement of angles.* Since the idea of a right angle is simple, and all right angles are equal, it is conveniently taken as a standard by which to measure the magnitudes of other angles. But since from the magnitude of a right angle most of the angles we have to deal with would be represented by fractions or decimals, it is found convenient to divide it.

In England we divide the right angle into 90 equal parts called degrees; each degree is divided into 60 minutes; each minute into 60 seconds. Any angle is then measured by the number of degrees, minutes, seconds, and decimal parts of a second it contains; an angle containing 27 degrees, 13 minutes, 24.53 seconds is written thus, $27^{\circ}, 13', 24''.53$.

In France the decimal system is adopted. A right angle contains 100 grades, a grade 100 minutes, a minute 100 seconds; an angle containing 33 grades, 27 minutes, 45.5 seconds is written thus, $33^{g}, 27^{\backprime}, 45^{\backprime\backprime}.5$. The advantage of this system is, that any angle can be reduced to the decimal of a grade at once, and *vice versâ*. Thus $33^{g}, 27^{\backprime}, 45^{\backprime\backprime}.5 = 33^{g}.27455$, and $27^{g}.35679 = 27^{g}, 35^{\backprime}, 67^{\backprime\backprime}.9$.

5. As an illustration of the use of different units, we will shew how to change our unit from degrees to grades, and *vice versâ*.

Let D be the number of degrees in any angle ACB, G the number of grades in the same angle; then since the number of degrees in two angles must bear to one another the same ratio which the angles themselves do, we have

$$\frac{\text{angle } ACB}{\text{a right angle}} = \frac{D}{90};$$

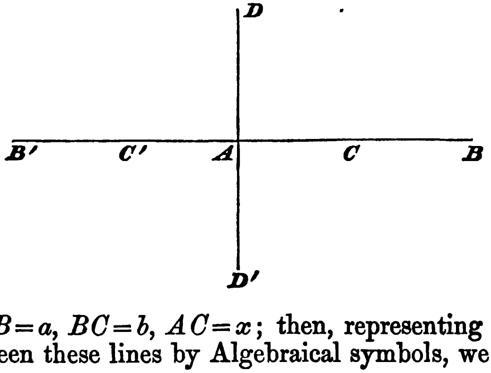
for a like reason

$$\frac{\text{angle } ACB}{\text{a right angle}} = \frac{G}{100};$$

Hence $\frac{D}{90} = \frac{G}{100}$: from which equation, if we know the number of degrees, we can find the number of grades, and *vice versâ*.

6. *Use of signs + and – to represent contrariety of position.* In applying Algebra to the solution of Arithmetical Problems, we frequently meet with negative results, which require interpretation in each particular case; we shall meet with similar results in Trigonometry; and the question arises, how are we to interpret a negative quantity when it represents a line or angle?

Suppose BAB', DAD' to be two lines at right angles to one another and fixed in position.

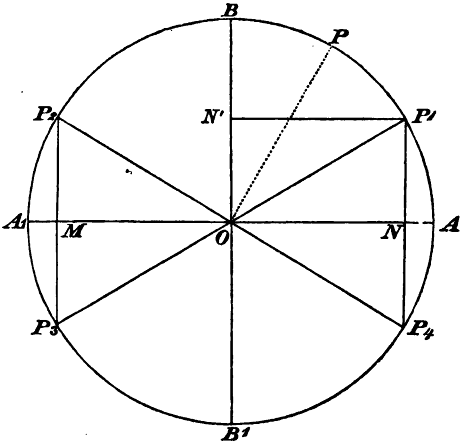


Let $AB = a$, $BC = b$, $AC = x$; then, representing the relation between these lines by Algebraical symbols, we have

$$x = a - b.$$

Now so long as $b < a$, x is positive, and C lies to the right of A: but if $b > a$, x is negative, and C lies to the left of A: that is, in using Algebraical symbols, if we arrive at a negative quantity representing a line, the natural interpretation appears to be, that the line must be considered as measured in an opposite direction to what it would be if the quantity representing it were positive. Hence we are led to the following convention: any line measured along AB or parallel to it, is said to be a positive line, and any line measured along AB' or parallel to it is said to be a negative line, and the symbol representing it must have the sign $-$ placed before it. So also lines measured along AD are positive, and along AD' negative. Lines measured in any other direction, not parallel to either of these two, will be considered positive. The advantage of these conventions will appear to the student as he proceeds.

7. *Magnitudes of angles unlimited in Trigonometry.*



In Euclid the magnitude of an angle is the absolute amount of the inclination of the two lines which contain it, to each other; this amount can never exceed two right angles. In

Trigonometry the idea of an angle is extended. Suppose AOA' to be a line fixed in space, and OP to be a line revolving from its original position OA towards OB, and tracing out the angle AOP: till OP reaches OA', this angle does not differ from Euclid's idea of an angle, because it is less than two right angles; but when OP has passed OA', and arrived at such a position as OP_3, the Trigonometrical idea of the angle differs from Euclid's: Euclid would now measure the angle AOP_3 by the amount of inclination of OP_3 to OA, that is, he would consider it as less than two right angles; but in Trigonometry we consider it as the angle actually described by the revolving line OP, that is, as $180^\circ + \angle A'OP_3$. So also when the revolving line reaches OP_4, the Trigonometrical angle AOP_4 will be $270^\circ + \angle B'OP_4$; and when OP has passed its initial position OA and reached OP_1 a second time, the Trigonometrical angle AOP_1 will be $360^\circ + \angle AOP_1$. The same plan for estimating the value of the angle extends to any number of revolutions of OP, and thus the magnitude of an angle in Trigonometry is unlimited.

If the line OP on starting from OA had revolved in the opposite direction towards OB', then in accordance with the principle laid down in Art. 6, we should have considered the angle described as a negative angle; and we have the same convention for angles that we have for lines, viz., that angles measured upwards from OA towards OB are positive angles, and angles measured downwards from OA to OB' are negative angles, and have the sign $-$ placed before the symbol representing them.

Obs. Also the space between OA and OB is called the first quadrant, and when OP lies between OA and OB, the angle AOP is said to be an angle in the first quadrant. So also angles AOP_2, AOP_3, AOP_4 are angles in the second, third, and fourth quadrants respectively.

8. *Complement; Supplement.* The complement of an angle is that angle which must be added to it to make it equal to a right angle. Thus the complement of

$$30^\circ \text{ is } 60^\circ, \text{ of } 120^\circ \text{ is } -30^\circ, \text{ of } -75^\circ \text{ is } 165^\circ.$$

The supplement of an angle is that angle which must be added to it to make it equal to two right angles. Thus the supplement of

$$30^\circ \text{ is } 150^\circ, \text{ of } 220^\circ \text{ is } -40^\circ, \text{ of } -75^\circ \text{ is } 255^\circ.$$

It is evident that in a right-angled triangle, one of the acute angles is the complement of the other; and in any triangle any one of the angles is supplementary to the sum of the other two.

9. *Trigonometrical Ratios.* Having now shewn how lines and angles are to be measured in Trigonometry, and the interpretation which is to be given to the signs + and − when applied to quantities that represent them, we will proceed to shew how lines are connected with angles: this is done by means of what are called Trigonometrical Ratios.

Let the revolving line OP (fig. Art. 7) have come into any position OP_1, OP_2, ..., and from P_1, P_2, ..., drop the perpendiculars P_1N, P_2M, ... upon the initial line OA, or OA produced backwards to A'; the triangles P_1NO, P_2MO, ... will be right-angled triangles. We then have the following definitions:

The ratio of the perpendicular to the hypothenuse is called the *sine* of the angle described: that is,

$$\frac{P_1N}{OP_1}, \ \frac{P_2M}{OP_2}, \dots$$

are the sines of the angles AOP_1, AOP_2, ... respectively.

The ratio of the base to the hypothenuse is called the *cosine* of the angle, that is,

$$\frac{ON}{OP_1}, \ \frac{OM}{OP_2}, \dots$$

are the cosines of AOP_1, AOP_2, ... respectively.

The ratio of the perpendicular to the base, is called the *tangent* of the angle, that is,

$$\frac{P_1N}{ON}, \quad \frac{P_2M}{OM}, \ldots.$$

are the tangents of AOP_1, AOP_2, ... respectively.

The ratio of the base to the perpendicular is called the *cotangent* of the angle, that is,

$$\frac{ON}{P_1N}, \quad \frac{OM}{P_2M}, \ldots.$$

are the cotangents of AOP_1, AOP_2, ... respectively.

The ratio of the hypothenuse to the base is called the *secant* of the angle, that is,

$$\frac{OP_1}{ON}, \quad \frac{OP_2}{OM}, \ldots.$$

are the secants of AOP_1, AOP_2, ... respectively.

The ratio of the hypothenuse to the perpendicular is called the *cosecant* of the angle, that is,

$$\frac{OP_1}{P_1N}, \quad \frac{OP_2}{P_2M}, \ldots.$$

are the cosecants of AOP_1, AOP_2, ... respectively.

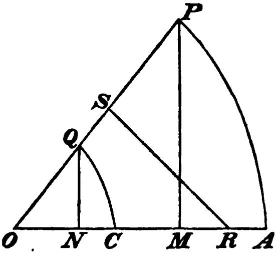
These definitions are quite independent of the direction and number of the revolutions of OP about O; that is, they apply to negative angles, and to angles greater than four right angles.

The Trigonometrical Ratios of any angle α (that is, of any angle which contains α degrees, grades, or any other unit we may employ) are usually written

$$\sin\alpha, \cos\alpha, \tan\alpha, \cot\alpha, \sec\alpha, \operatorname{cosec}\alpha.$$

10. The question here naturally arises, are these Trigonometrical ratios definite quantities for any given value of the angle, or do they depend upon the magnitude of the revolving line OP?

Let AOP be any angle, i. e. let OP be any position whatever of the revolving line. Take Q any point in OP, or OP produced; draw PM, QN perpendicular to OA.



Then by similar triangles PMO, QNO,

$$QN : ON : OQ = PM : OM : OP.$$

Hence the Trigonometrical Ratios of the angle AOP are the same whether we consider OQ or OP the revolving line.

So also if RS is perpendicular to OP, we have

$$RS : OS : OR = PM : OM : OP.$$

Hence we conclude, that, *disregarding signs*, the values of the Trigonometrical Ratios are quite independent of the magnitude of the revolving line, and depend only on the absolute inclination of the two lines containing the angle to each other; and so for a given value of the angle are determinate.

The converse of this proposition (viz. that for a given value of the Trigonometrical Ratio, the angle is determinate) is only true with certain limitations as will appear hereafter.

Obs. The necessity of the proviso *disregarding signs* will appear on reading the next section.

11. Let the angle AOP (fig. Art. 7) contain α degrees, then P_1OB, and OP_1N (since they either of them with AOP_1 make a right angle) contain $90-\alpha$ degrees each. Drawing P_1N' perpendicular to OB, we have by our definitions

$$\sin P_1OB = \frac{P_1N'}{OP_1} = \frac{ON}{OP_1} = \cos AOP_1,$$

$$\text{or } \sin(90^\circ - \alpha) = \cos\alpha.$$

Again, $$\tan P_1OB = \frac{P_1N'}{ON'} = \frac{ON}{P_1N} = \cot AOP_1,$$

$$\text{or } \tan(90^\circ - \alpha) = \cot\alpha,$$

$$\text{so also } \sec(90^\circ - \alpha) = \operatorname{cosec}\alpha,$$

and these equations may be proved to exist for any position of the revolving line OP.

It is from these relations, viz. that the *co*sine, *co*tangent, *co*-secant of an angle, are respectively the same as the sine, tangent, and secant of the complement, that they derive their names.

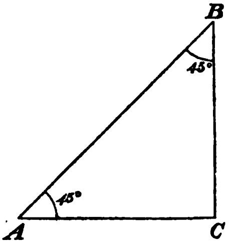
From the same relations we also have

$$\sin\alpha = \cos(90^\circ - \alpha),$$
$$\tan\alpha = \cot(90^\circ - \alpha),$$
$$\sec\alpha = \operatorname{cosec}(90 - \alpha).$$

Hence we can always express the Trigonometrical Ratio of the complement of an angle in terms of some Trigonometrical Ratio of the simple angle.

12. *The Trigonometrical Ratios of* 30°, 45°, 60°. The calculation of the values of the Trigonometrical Ratios of all angles is a matter of considerable labour, and the processes employed to do it will not be described in this treatise, but their values may be easily found in some particular cases.

Let ABC be an isosceles triangle having a right angle at C; then CAB = half a right angle $= 45^\circ$.



Let $AC = x = BC$,

then $AB^2 = AC^2 + BC^2 = 2x^2$,

or $AB = x\sqrt{2}$,

and we have

$$\sin 45^\circ = \frac{BC}{AB} = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}},$$

$$\cos 45^\circ = \frac{AC}{AB} = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}},$$

$$\tan 45^\circ = \frac{BC}{AC} = \frac{x}{x} = 1,$$

$$\cot 45^\circ = \frac{AC}{BC} = \frac{x}{x} = 1,$$

$$\sec 45^\circ = \frac{AB}{AC} = \frac{x\sqrt{2}}{x} = \sqrt{2},$$

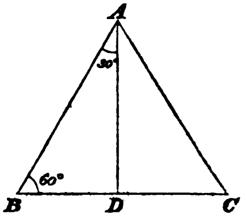
$$\operatorname{cosec} 45^\circ = \frac{AB}{BC} = \frac{x\sqrt{2}}{x} = \sqrt{2}.$$

Again, let ABC be an equilateral triangle, then angle $ABC = \frac{180^\circ}{3} = 60^\circ$. Let AD be perpendicular to BC, then angle $BAD = 30^\circ$. Let $BD = x$, $AB = 2x$; then

$$AB^2 = AD^2 + BD^2,$$

$$\text{or } 4x^2 = AD^2 + x^2,$$

and therefore $AD = x\sqrt{3}$; and we have,



$$\sin 60^\circ = \cos 30^\circ = \frac{AD}{AB} = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2},$$

$$\cos 60^\circ = \sin 30^\circ = \frac{BD}{AB} = \frac{x}{2x} = \frac{1}{2},$$

$$\tan 60^\circ = \cot 30^\circ = \frac{AD}{BD} = \frac{x\sqrt{3}}{x} = \sqrt{3},$$

$$\cot 60^\circ = \tan 30^\circ = \frac{BD}{AD} = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}},$$

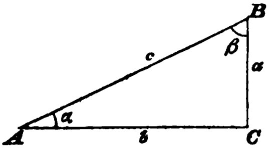
$$\sec 60^\circ = \operatorname{cosec} 30^\circ = \frac{AB}{BD} = \frac{2x}{x} = 2,$$

$$\operatorname{cosec} 60^\circ = \sec 30^\circ = \frac{AB}{AD} = \frac{2x}{x\sqrt{3}} = \frac{2}{\sqrt{3}}.$$

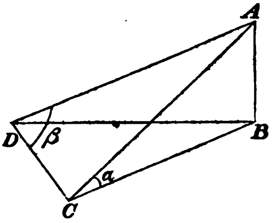
Thus we have found the values of all the Trigonometrical Ratios of 30°, 45°, and 60°.

13. We will now assume that we know the values of the Trigonometrical Ratios of all angles, as well as of 30°, 45°, and 60°, and shew how they may be used to solve many interesting problems.

Let ABC be a right-angled triangle, C the right angle, and let angles CAB, CBA contain α^0, β^0 respectively. Let the lengths of the sides opposite to A, B, C be a, b, c respectively. Then $\sin\alpha = \frac{a}{c}$, or $a = c\sin\alpha$; $\tan\alpha = \frac{a}{b}$, or $a = b\tan\alpha$; $\cos\alpha = \frac{b}{c}$, or $b = c\cos\alpha$; and similarly other relations may be found to exist between the sides and angles of a right-angled triangle. The student will best learn by practice to apply these relations to solve problems in each particular case; the following may be taken as an example.



A person on one side of a river at C wishes to determine the height of a tower AB on the other side. He observes by an instrument made for the purpose, the angle which CA makes with the horizontal line CB. (Let this be α). He then places a mark at C, and measures the distance CD (a feet) at right angles to CB, and when at D he observes the angle ADC (β suppose). He then calculates the height of the tower thus; ACD is a right-angled triangle, and therefore



$$AC = CD\tan ADC, \quad \text{or} \quad AC = a\tan\beta.$$

So also ABC is a right-angled triangle, and therefore

$$AB = AC\sin ACB = AC\sin\alpha.$$

But
$$AC = a\tan\beta,$$
$$\therefore AB = a\tan\beta\sin\alpha.$$

Hence since we know the values of a, $\tan\beta$, $\sin\alpha$, we can calculate the height of the tower.

Ex. $a = 50$ yards, $\beta = 45^\circ$, $\alpha = 30^\circ$,

$$\text{height of tower} = 50 \times 1 \times \tfrac{1}{2} = 25 \text{ yards}.$$

Similarly we might find the breadth of the river.

14. The two principal instruments by which observations are taken in surveying are the sextant and the theodolite. The construction of these instruments depends on certain properties of light, the discussion of which falls under the science of optics. By the sextant we are enabled to observe the angle which any two objects subtend at the eye of the observer, as for instance the angle ADC in the case above.

By the theodolite we can observe the elevation of any object, that is to say, the angle which the line joining the object and the eye of the observer makes with the horizontal plane, as the angle ACB in the case above. We can also observe the angle which two objects in the same horizontal plane subtend at the eye of the observer; the sextant might do this, if we knew that the objects were in the same horizontal plane; the theodolite tells us that they are, as well as the amount of the angle they subtend. When an object is lower than the observer, the angle the line joining it makes with the horizontal plane is called its depression.

EXAMPLES. (A).

1. Find the number of degrees in the angle $17^g, 34^{`}, 27^{``}$.

We have $17^g, 34^{`}, 27^{``} = 17^g.3427$,

and $\dfrac{D}{90} = \dfrac{G}{100}$, or $D = \dfrac{9}{10} \times 17.3427 = 15.60843$ degrees

$= 15^\circ, 36', 30''.348$. *Ans.* $15^\circ, 36', 30''.348$.

2. Construct the angle whose sine is $\frac{1}{\sqrt{3}}$.

Take BD (fig. Art. 12) $= x$, then $AD = x\sqrt{3}$; with centre B, distance AD describe a circle meeting AD in E, join BE, then

$$\sin BED = \frac{BD}{BE} = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}},$$

and the angle BED has been found as required.

3. If a line of 10 feet be represented by 3, what is the unit of length?

4. How is 37 yards, 2 feet, 8 inches represented, when the unit of length is 4 inches? *Ans.* By 341.

5. If the angle of an equilateral triangle be taken as the unit of measurement, how many degrees will be contained in the angle $.\dot{3}$?

6. Find the number of grades in $27^\circ.35'.24''$.

Ans. $30^g.6\dot{5}$.

7. What is the supplement of $227^g, 35^`, 46^{``}$?

8. The complement of $45^\circ \pm \alpha$ is $45^\circ \mp \alpha$.

9. The difference of two acute angles of a right-angled triangle is δ°, find the angles.

10. The vertical angle of an isosceles triangle is α°, find the angles at the base.

11. The length of a person's shadow is twice his height, find the altitude of the sun, having given $\tan 26^\circ.34' = .5$.

12. Shew that $(\sin 60^\circ - \sin 45^\circ)(\cos 30^\circ + \cos 45^\circ) = \sin^2 30^\circ$.

13. Shew that $(\sin 30^\circ + \cos 30^\circ)(\sin 60^\circ - \cos 60^\circ) = \cos 60^\circ$.

14. Shew that $\frac{\sin 45^\circ - \sin 30^\circ}{\sin 45^\circ + \sin 30^\circ} = (\sec 45^\circ - \tan 45^\circ)^2$.

15. Determine the height of an object whose elevation is 40°, the observer being 140 feet distant, and his eye 5 feet from the ground, having given that $\tan 40^\circ = .85$.

Ans. 124 feet.

16. Construct the angle whose tangent is $\sqrt{2}$.

17. Construct the angle whose sine is $\frac{\sqrt{2}-1}{2}$.

18. A ladder of length l is placed against a wall so that the angle it makes with the ground is double that which it makes with the wall: find how far from the wall the foot of the ladder is.

Ans. $\frac{l}{2}$.

19. The town C is halfway between the towns D and E; and the towns C, E, F are equidistant from each other. If distance from D to E is 12 miles, find the distance from D to F.

Ans. $6\sqrt{3}$ miles.

20. A ladder 20 feet long just reaches the top of a wall when its foot is 13 feet from the foot of the wall; shew that when its foot is 5 feet from the wall, the ladder projects 4 feet beyond the top of the wall.

21. In a right-angled triangle the lengths of lines drawn from the acute angles to the middle points of the opposite sides are d, d' respectively. Find the sides of the triangle.

Ans. $\frac{2}{\sqrt{15}}\sqrt{(4d^2 - d'^2)}, \quad \frac{2}{\sqrt{15}}\sqrt{(4d'^2 - d^2)}$.

22. A person travelling southwards observes two objects towards the S.E. After 8 miles travelling, one of them is N.E., and the other E. Their distances from him are then $4\sqrt{2}$ miles, and 8 miles respectively.

23. Whilst sailing due West, I observe two ships at anchor directly North of me: after sailing 6 miles the directions of the ships make angles 60°, 30° with my course respectively. The distance between them is $4\sqrt{3}$ miles.

24. A staff 1 foot long stands on the top of a tower 200 feet high. Shew that the angle it subtends at a place 100 feet from the foot of the tower is $6'$.

Given $\tan 63^\circ.27' = 2$, $\tan 63^\circ.33' = 2.01$.

25. The angles of depression of the top and bottom of a column observed from a tower 108 feet high are 30°, 60° respectively. Shew that the height of the column is 72 feet.

26. From the top of a column 100 feet high, the angles of depression of two objects in a line with the column, and in the horizontal plane on which the column stands, are observed to be 30° and 60°. The distance between them is $\frac{200}{\sqrt{3}}$ feet.

27. The length of a road in which the ascent is 1 foot in 5 from the foot of a hill to the top is a mile and two-thirds: what will be the length of a zigzag road in which the ascent is 1 foot in 12? *Ans.* 4 miles.

EXAMPLES. (B).

1. The same line is represented by m and n in two systems of measurement, compare the units of length in the two systems.

2. What is the greatest unit of length with which .625 feet may be represented by an integer?

3. Find the number of French minutes in one English minute, and the number of French seconds in one English second. *Ans.* $1^{\backprime}.\dot{8}5\dot{1}$, $3^{\backprime\backprime}.\dot{0}8641975\dot{3}$.

4. The angles of a triangle are in $A.P$; shew that one of them must equal 60°.

5. The supplement of one angle of a triangle is double the complement of another, and triple that of the third: find the angles. *Ans.* $81\frac{9}{11}$, $40\frac{10}{11}$, $57\frac{3}{11}$ degrees.

6. A person observes the angular elevation of a column; after approaching a feet, he finds its elevation doubled; again approaching b feet it is again doubled. Shew that the second point of observation is $\frac{a^2}{2b}$ feet from the foot of the tower.

7. Construct the angle whose tangent is $\sqrt{5}-1$.

8. At the foot of a mountain the elevation of its summit is 45°. After ascending 1 mile up a slope of 30° its elevation is found to be 60°. The height of the mountain is $\frac{\sqrt{3}+1}{2}$ miles.

9. A person at the edge of a river observes the elevation α of a tower on the other side; on retreating a feet he finds its elevation to be β: the height of the tower is $\frac{a\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$, the breadth of river is $\frac{a\tan\beta}{\tan\alpha-\tan\beta}$.

10. A rock is observed from a ship to bear N.N.W; after sailing 10 miles in direction E.N.E the rock is due West: its distance from the ship at the first observation was $10(\sqrt{2}-1)$ miles. Given $\tan 22\frac{1}{2}^\circ=\sqrt{2}-1$.

11. Two persons A and B start from two points distant 400 yards. B starts at right angles to the line joining the two points at the rate of 90 yards a minute. A starts in a direction to catch B as soon as possible at the rate of 150 yards a minute; find how long he will be before he catches him, and the direction in which he will walk, having given $\sin 36^\circ.\ 53'=\frac{3}{5}$. *Ans.* 3 minutes, 20 seconds.

12. A tower is situated at the top of a hill whose inclination is 30°. The angle subtended by the tower at the foot of the hill is 15°; and on walking a yards up the hill it is found to be 30°. The height of the tower is $\frac{a}{\sqrt{3}}$ feet.

13. From two points in the diameter of a circle produced tangents are drawn to the circle. Given the distance c between the points, and the inclinations α, β of the tangents to the diameter, shew that the radius of the circle

$$=\frac{c}{\operatorname{cosec}\beta-\operatorname{cosec}\alpha}.$$

14. A person travelling along a road observes the elevation α of a tower the nearest distance (a) of which from the road is known; at the same time he observes the angular distance β of the top of the tower from an object in the road. The height of the tower is $\dfrac{a\sin\alpha}{\sqrt{(\cos^2\alpha-\cos^2\beta)}}$.

15. At three positions in the same horizontal plane distant from each other 60, 80, 100 feet respectively, the elevation of a tower is observed to be 45°; find its height.

Ans. 50 feet.

16. Two spectators at two stations, distant $2a$ from each other, observe the elevation of a kite to be α at each station, and the angle subtended by the kite and the other station to be β; shew that the height of the kite is $a\sec\beta\sin\alpha$.

17. A person standing on the top A of a light-house AB, of known height 300 feet, observes a ship sailing from C to D in a straight line: he knows that CD is perpendicular to the plane ACB, and he observes the angle $CAD=30°$, $BAD=60°$. Shew that $CD=300$ feet; $BD=300\sqrt{3}$ feet.

18. The altitude of the sun is 45°, and it is at a point of the compass 60° from the south. The breadth of the shadow of a south wall is one-half its height.

19. The angular altitude and breadth of a cylindrical tower are observed to be α and β respectively; but at a point a feet nearer the foot of the tower they are α', β' respectively: find the height and radius of the tower.

Ans. Height $=\dfrac{a}{\cot\alpha-\cot\alpha'}$, radius $=\dfrac{a}{\operatorname{cosec}\dfrac{\beta}{2}-\operatorname{cosec}\dfrac{\beta'}{2}}$.

SECTION II.

TRIGONOMETRICAL RATIOS OF ANGLES GREATER THAN 90°. RELATIONS AMONGST THE RATIOS. CHANGE IN SIGN AND MAGNITUDE OF THE RATIOS THROUGH THE FOUR QUADRANTS. CHANGE FROM POSITIVE TO NEGATIVE INFINITY. OTHER TRIGONOMETRICAL RATIOS.

15. We now proceed to express the Trigonometrical Ratios of all angles in terms of Trigonometrical ratios of positive angles less than a right angle.

We will first find the Trigonometrical Ratios of the supplement of an angle.

Let $A'OP_2 = AOP_1$ absolutely (fig. Art. 7): then AOP_2 is the supplement of AOP_1 or α, that is, $AOP_2 = 180^\circ - \alpha$.

Let $ON = x, \quad P_1N = y, \quad OP_1 = r,$

then $OM = -x, \quad P_2M = y, \quad OP_2 = r.$

Hence

$$\sin(180^\circ - \alpha) = \frac{y}{r} = \sin\alpha,$$

$$\cos(180^\circ - \alpha) = \frac{-x}{r} = -\cos\alpha,$$

$$\tan(180^\circ - \alpha) = \frac{y}{-x} = -\tan\alpha,$$

$$\cot(180^\circ - \alpha) = \frac{-x}{y} = -\cot\alpha,$$

$$\sec(180^\circ - \alpha) = \frac{r}{-x} = -\sec\alpha,$$

$$\text{cosec}(180^\circ - \alpha) = \frac{r}{y} = \text{cosec}\,\alpha;$$

which equations give the Trigonometrical Ratios of the supplement of an angle, in terms of the Trigonometrical Ratios of the simple angle.

We may observe that they all change in sign except the sine and cosecant.

16. We will next find the Trigonometrical Ratios of a negative angle.

Let $AOP_4 = AOP_1$ absolutely: then $AOP_4 = -\alpha$;

also $ON = x, \quad P_4N = -y, \quad OP_4 = r.$

Hence
$$\sin(-\alpha) = \frac{-y}{r} = -\sin\alpha,$$
$$\cos(-\alpha) = \frac{x}{r} = \cos\alpha,$$
$$\tan(-\alpha) = \frac{-y}{x} = -\tan\alpha,$$
$$\cot(-\alpha) = \frac{x}{-y} = -\cot\alpha,$$
$$\sec(-\alpha) = \frac{r}{x} = \sec\alpha,$$
$$\operatorname{cosec}(-\alpha) = \frac{r}{-y} = -\operatorname{cosec}\alpha;$$

which equations give the Trigonometrical Ratios of a negative angle in terms of those of the positive angle of the same absolute magnitude.

We may observe that they all change in sign except the cosine and secant.

17. Let $A'OP_3 = AOP_1 = \alpha$, then $AOP_3 = 180^\circ + \alpha$,

also $OM = -x, \quad P_3M = -y, \quad OP_3 = r.$

Hence
$$\sin(180^\circ + \alpha) = \frac{-y}{r} = -\sin\alpha,$$
$$\cos(180^\circ + \alpha) = \frac{-x}{r} = -\cos\alpha,$$

$$\tan(180^\circ+\alpha)=\frac{-y}{-x}=\tan\alpha,$$

$$\cot(180^\circ+\alpha)=\frac{-x}{-y}=\cot\alpha,$$

$$\sec(180^\circ+\alpha)=\frac{r}{-x}=-\sec\alpha,$$

$$\operatorname{cosec}(180^\circ+\alpha)=\frac{r}{-y}=-\operatorname{cosec}\alpha.$$

Here all the ratios change in sign except the tangent and cotangent.

18. It is evident that, when the revolving line OP has reached OP, the second time, that is, has described the angle $(360^\circ+\alpha)$, all the Trigonometrical Ratios are the same as those of the angle α, for the triangle which determines their values is identical in the two cases. The same remark applies to any number of revolutions, so that we may say generally that the Trigonometrical Ratios of $(n\,.\,360^\circ+\alpha)$ are equal severally to those of α.

19. The formulæ in Articles 12, 15, 16, 17, have been proved for the case in which α is less than 90°; that they are true for all values whatever of α the student may satisfy himself by drawing figures to represent the different cases: for a more logical proof that these forms are true for all angles whatever he must be content to wait till he is further advanced, as the reasoning is of too subtle a nature for the present.

These formulæ may also be classed as follows:

$$\sin\alpha=\sin(180^\circ-\alpha)=-\sin(180^\circ+\alpha)=-\sin(-\alpha)$$
$$=\sin(n\,.\,360^\circ+\alpha),$$

$$\cos\alpha=-\cos(180^\circ-\alpha)=-\cos(180^\circ+\alpha)=\cos(-\alpha)$$
$$=\cos(n\,.\,360^\circ+\alpha),$$

$$\tan\alpha=-\tan(180^\circ-\alpha)=\tan(180^\circ+\alpha)=-\tan(-\alpha)$$
$$=\tan(n\,.\,360^\circ+\alpha),$$

$$\cot\alpha = -\cot(180^\circ-\alpha) = \cot(180^\circ+\alpha) = -\cot(-\alpha)$$
$$= \cot(n\,.\,360^\circ+\alpha),$$

$$\sec\alpha = -\sec(180^\circ-\alpha) = -\sec(180^\circ+\alpha) = \sec(-\alpha)$$
$$= \sec(n\,.\,360^\circ+\alpha),$$

$$\operatorname{cosec}\alpha = \quad \operatorname{cosec}(180^\circ-\alpha) = -\operatorname{cosec}(180^\circ+\alpha) = -\operatorname{cosec}(-\alpha)$$
$$= \operatorname{cosec}(n\,.\,360^\circ+\alpha).$$

We obtain also from the above formulæ the signs which the Trigonometrical Ratios of any angles must have according to the quadrant in which their bounding line is situated.

20. We are now in a position to express the Trigonometrical Ratios of any angle whatever in terms of the ratios of a positive angle less than 90°.

If the angle is negative make it positive by formulæ in Art. 16.

If the angle is greater than 360°, suppress 360° as often as necessary to make it less than 360° by Art. 18.

If the angle is now greater than 180°, suppress 180° by Art. 17.

If the angle is now greater than 90° take its supplement by Art. 15.

When the angle is negative labour is sometimes saved by adding 360° once or oftener, till the angle is positive, instead of using Art. 16.

Ex. $\tan 995^\circ = \tan 275^\circ = \tan 95^\circ = -\tan 85^\circ,$

$\operatorname{cosec}(-995^\circ) = -\operatorname{cosec} 995^\circ = -\operatorname{cosec} 275^\circ = \operatorname{cosec} 95^\circ = \operatorname{cosec} 85^\circ,$

or thus, $\operatorname{cosec}(-995^\circ), = \operatorname{cosec}(1080^\circ - 995^\circ) = \operatorname{cosec} 85^\circ.$

21. From the formulæ in Arts. 15, 16, 17, 18 we deduce the following, which are true for all values of the angle,

$$\sin\alpha \operatorname{cosec}\alpha = 1,$$

$$\cos\alpha \sec\alpha = 1,$$

$$\tan\alpha \cot\alpha = 1.$$

Also in the first quadrant $\tan\alpha = \dfrac{y}{x} = \dfrac{\frac{y}{r}}{\frac{x}{r}} = \dfrac{\sin\alpha}{\cos\alpha}$,

in the second quadrant $\tan\alpha = \dfrac{y}{-x} = \dfrac{\frac{y}{r}}{-\frac{x}{r}} = \dfrac{\sin\alpha}{\cos\alpha}$,

in the third quadrant $\tan\alpha = \dfrac{-y}{-x} = \dfrac{-\frac{y}{r}}{-\frac{x}{r}} = \dfrac{\sin\alpha}{\cos\alpha}$,

in the fourth quadrant $\tan\alpha = \dfrac{-y}{x} = \dfrac{-\frac{y}{r}}{\frac{x}{r}} = \dfrac{\sin\alpha}{\cos\alpha}$.

Hence always $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$;

and also $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$.

Also since $x^2 + y^2 = r^2$, (Euclid I. 47)

divide by r^2, and we have

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1,$$

or $\cos^2\alpha + \sin^2\alpha = 1$;

whence $\sin\alpha = \pm\sqrt{(1 - \cos^2\alpha)}$;

$\cos\alpha = \pm\sqrt{(1 - \sin^2\alpha)}$:

divide by x^2, and we have

$$\frac{r^2}{x^2} = 1 + \frac{y^2}{x^2},$$

or $$\sec^2 \alpha = 1 + \tan^2 \alpha;$$

whence $$\sec \alpha = \pm \sqrt{(1 + \tan^2 \alpha)};$$

$$\tan \alpha = \pm \sqrt{(\sec^2 \alpha - 1)}:$$

divide by y^2, and we have,

$$\frac{r^2}{y^2} = 1 + \frac{x^2}{y^2},$$

or $$\operatorname{cosec}^2 \alpha = 1 + \cot^2 \alpha;$$

whence $$\operatorname{cosec} \alpha = \pm \sqrt{(1 + \cot^2 \alpha)};$$

$$\cot \alpha = \pm \sqrt{(\operatorname{cosec}^2 \alpha - 1)}:$$

Also we have $\sin \alpha = \frac{y}{r} = \pm \frac{y}{\sqrt{(x^2 + y^2)}} = \pm \frac{\frac{y}{x}}{\sqrt{\left(1 + \frac{y^2}{x^2}\right)}}$

$$= \pm \frac{\tan \alpha}{\sqrt{(1 + \tan^2 \alpha)}};$$

so also $$\cos \alpha = \pm \frac{1}{\sqrt{(1 + \tan^2 \alpha)}};$$

$$\sin \alpha = \pm \frac{1}{\sqrt{(1 + \cot^2 \alpha)}};$$

$$\cos \alpha = \pm \frac{\cot \alpha}{\sqrt{(1 + \cot^2 \alpha)}}.$$

The upper or lower sign must be used in these forms according to the quadrant in which the angle lies.

Obs. Some of these expressions are really included in the others, for they are derived from the same relation $x^2 + y^2 = r^2$. We may also put $90 - \alpha$ for α in any of these relations, and they must still remain true; thus the relation

$$\sin \alpha = \pm \frac{\tan \alpha}{\sqrt{(1+\tan^2 \alpha)}}$$

becomes $$\sin(90-\alpha) = \pm\frac{\tan(90-\alpha)}{\sqrt{\{1+\tan^2(90-\alpha)\}}};$$

or, $$\cos\alpha = \pm\frac{\cot\alpha}{\sqrt{(1+\cot^2\alpha)}},$$

as we obtained before. In fact, whatever *general* relation holds between any Trigonometrical Ratios, must necessarily hold if the complementary or supplementary ratios are substituted for them.

22. We stated above (Art. 10) that when we know the value of any of the Trigonometrical Ratios, the sine for instance, we only know the value of the corresponding angle within certain limitations: we can now more fully explain this. From the preceding Articles we may observe, that whenever the revolving line OP (fig. Art. 7) has come into either the position OP_1, or OP_2, the value of the sine of the angle described is the same and of the same sign. Hence, if the value and sign of the sine is given, and the sine of AOP_1 is equal to this given value and of the same sign, all we know about the angle whose sine is given is, that it is bounded by one of the lines OP_1, OP_2; but of the number of revolutions the line OP has taken, we are quite ignorant. Similar remarks will apply to all the Trigonometrical Ratios.

23. As an illustration of the principles laid down in the preceding Articles, we will trace the changes in sign and magnitude of the Trigonometrical Ratios as the angle (fig. Art. 7) AOP or α passes through all values from 0° to 360°. Let

$$ON = x,\quad PN = y,\quad OP = r,$$

for all positions of OP: then r is always positive and never less than x and y, and therefore the sine and cosine are never greater than 1, and the secant and cosecant never less than 1: and since x and y may have any ratio to each other, the tangent and cotangent may have any values whatever.

When OP coincides with OA, $x = r$, $y = 0$.

Hence, $\sin 0^\circ = \frac{0}{r} = 0$; $\tan 0^\circ = \frac{0}{r} = 0$; $\sec 0^\circ = \frac{r}{r} = 1$;

$$\cos 0^\circ = \frac{r}{r} = 1;\ \cot 0^\circ = \frac{r}{0} = \infty;\ \operatorname{cosec} 0^\circ = \frac{r}{0} = \infty:$$

As *OP* revolves from *OA* to *OB*, x diminishes, y increases, and both are positive; hence the sine, tangent, secant increase, the cosine, cotangent, cosecant diminish, and they are all positive.

When *OP* coincides with *OB*, $x = 0$, $y = r$;

Hence, $\sin 90^\circ = \frac{r}{r} = 1$; $\tan 90^\circ = \frac{r}{0} = \infty$; $\sec 90^\circ = \frac{r}{0} = \infty$;

$$\cos 90^\circ = \frac{0}{r} = 0;\ \cot 90^\circ = \frac{0}{r} = 0;\ \operatorname{cosec} 90^\circ = \frac{r}{r} = 1.$$

As *OP* revolves from *OB* to *OA'*, x increases and is negative, y diminishes and is positive; hence the sine, tangent, secant diminish, the cosine, cotangent, cosecant increase; also the sine and cosecant are positive, the rest negative.

When *OP* coincides with *OA'*, $x = -r$, $y = 0$;

Hence,

$$\sin 180^\circ = \frac{0}{r} = 0;\ \tan 180^\circ = \frac{0}{-r} = 0;\ \sec 180^\circ = \frac{r}{-r} = -1;$$

$$\cos 180^\circ = \frac{-r}{r} = -1;\ \cot 180^\circ = \frac{-r}{0} = -\infty;\ \operatorname{cosec} 180^\circ = \frac{r}{0} = \infty.$$

As *OP* revolves from *OA'* to *OB'*, x diminishes and is negative, y increases and is negative; hence the sine, tangent, secant increase, and the cosine, cotangent, cosecant diminish, also the tangent and cotangent are positive, the rest negative.

When *OP* coincides with *OB'*, $x = -0$, $y = -r$;

Hence,

$$\sin 270^\circ = \frac{-r}{r} = -1;\ \tan 270^\circ = \frac{-r}{-0} = \infty;\ \sec 270^\circ = \frac{r}{-0} = -\infty;$$

$$\cos 270^\circ = \frac{-0}{r} = 0;\ \cot 270^\circ = \frac{-0}{-r} = 0;\ \operatorname{cosec} 270^\circ = \frac{r}{-r} = -1.$$

As OP revolves from OB' to OA, x increases and is positive, and y diminishes and is negative. Hence the sine, tangent, cotangent diminish, and the cosine, cotangent, cosecant increase; also the cosine and secant are positive, the rest negative.

When OP coincides with OA the second time, the values of the Trigonometrical Ratios become equal to those of 0°.

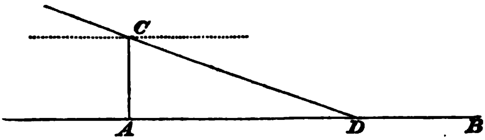
24. We have used above the relation $x = -0$, and so made

$$\sec 270^\circ = \frac{r}{x} = \frac{r}{-0} = -\infty,$$

instead of $+\infty$ which it would have been if we had put $x = 0$. The reason of this is, that x has been continually diminishing and negative, and therefore the secant has been a continually increasing negative quantity; and we have thought it better to say that as the angle reaches 270°, the secant becomes an infinitely great negative quantity. If however the line OP had approached the position OB' from OA, by revolving in the negative direction, the secant of the angle -90° would be $+\infty$.

Thus for the same position of OP, supposed to revolve in opposite directions to reach this position, we have two values of the secant of the angle of opposite signs. In fact, positive and negative infinity seem to have a near approach to each other, for we find the Ratios which become infinite always pass from one to the other. The case above and similar ones are only in accordance with our preceding formulæ; for since 270° and -90° are supplementary angles, it follows that their secants should be equal and of opposite signs.

The following illustration may help to remove the difficulty from the Student's mind.



Let CA be drawn perpendicular to AB a line of unlimited length. CD drawn through C to meet AB in D. Let $AD=x$. Now if angle ACD is made gradually to increase, the value of x increases without limit and is positive till ACD becomes a right angle. As soon as ACD becomes greater than a right angle, the value of x becomes suddenly an infinitely large negative quantity, and then diminishes to zero. The change in position of the point D for a small change in the revolving line, becomes infinitely great at this position of the line, though in general it is but gradual.

25. Besides the Trigonometrical Ratios above enumerated, the following are sometimes used.

The versine of the angle, which equals 1 – cosine, that is,

$$\text{vers}\,\alpha = 1 - \cos\alpha.$$

The coversine, which is the versine of the complement, that is,

$$\text{covers}\,\alpha = 1 - \cos(90-\alpha) = 1 - \sin\alpha.$$

The suversine, which is the versine of the supplement, that is,

$$\text{suvers}\,\alpha = 1 - \cos(180-\alpha) = 1 + \cos\alpha.$$

Also the chord of the angle, which is twice the sine of half the angle, that is,

$$\text{chd}\,\alpha = 2\sin\frac{\alpha}{2}.$$

26. The following notation is also used: if the sine of the angle α is a, that is, if $\sin\alpha = a$, then the angle whose sine is a is written $\sin^{-1}a$, that is, $\alpha = \sin^{-1}a$; and so for the other Trigonometrical Ratios.

EXAMPLES A.

1. If $\tan\alpha = \frac{b}{a}$, then $\sin\alpha = \frac{b}{\sqrt{(a^2+b^2)}}$, $\cos\alpha = \frac{a}{\sqrt{(a^2+b^2)}}$,

$$\text{for}\quad \sin\alpha = \frac{\tan\alpha}{\sqrt{(1+\tan^2\alpha)}} = \frac{\frac{b}{a}}{\sqrt{\left(1+\frac{b^2}{a^2}\right)}} = \frac{b}{\sqrt{(a^2+b^2)}},$$

$$\cos\alpha = \frac{1}{\sqrt{(1+\tan^2\alpha)}} = \frac{1}{\sqrt{\left(1+\frac{b^2}{a^2}\right)}} = \frac{a}{\sqrt{(a^2+b^2)}}.$$

2. If $\tan\theta = \dfrac{1+\tan\alpha}{1-\tan\alpha}$, find $\sin\theta$, $\cos\theta$,

$$\sin\theta = \frac{1+\tan\alpha}{\sqrt{\{2\,(1+\tan^2\alpha)\}}} = \frac{1+\tan\alpha}{\sec\alpha.\sqrt{2}} = \frac{1}{\sqrt{2}}\,(\cos\alpha+\sin\alpha),$$

$$\cos\theta = \frac{1-\tan\alpha}{\sqrt{\{2\,(1+\tan^2\alpha)\}}} = \frac{1}{\sqrt{2}}\,(\cos\alpha-\sin\alpha).$$

3. If $\sin\alpha = .\dot{3}$: shew that, $\tan\alpha = \frac{1}{4}\sqrt{2}$, $\cot\alpha = 2\sqrt{2}$, $\sec\alpha = \frac{3}{4}\sqrt{2}$.

4. If $\tan\alpha = .\dot{3}$, find $\sin\alpha$, $\cos\alpha$, $\operatorname{cosec}\alpha$, $\operatorname{vers}\alpha$.

Ans. $\dfrac{1}{10}\sqrt{10}$, $\dfrac{3}{10}\sqrt{10}$, $\sqrt{10}$, $\dfrac{\sqrt{10}-3}{\sqrt{10}}$.

Prove the truth of the following equations:

5. $(\cos\alpha\tan\alpha)^2 + (\sin\alpha\cot\alpha)^2 = 1.$

6. $\dfrac{\cos\alpha}{\sin\alpha\cot^2\alpha} = \tan\alpha.$

7. $\sec^2\alpha + \operatorname{cosec}^2\alpha = \sec^2\alpha\operatorname{cosec}^2\alpha.$

8. $\cot^2\alpha - \cos^2\alpha = \cos^2\alpha\cot^2\alpha.$

9. $\sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta = \sin^2\alpha - \sin^2\beta.$

10. $\cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta = \cos^2\alpha - \sin^2\beta.$

11. $\cos^4\alpha - \sin^4\alpha = \cos^2\alpha - \sin^2\alpha.$

12. $\tan\alpha + \cot\alpha = \sec\alpha\operatorname{cosec}\alpha.$

13. $\dfrac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta} = \tan\alpha\tan\beta.$

14. $\tan^2\alpha\tan^2\beta - 1 = \dfrac{\sin^2\alpha - \cos^2\beta}{\cos^2\alpha\cos^2\beta}$.

15. $\dfrac{1-\tan^2\alpha\tan^2\beta}{\tan^2\alpha\tan^2\beta} = \dfrac{\cos^2\alpha - \sin^2\beta}{\sin^2\alpha\sin^2\beta}$.

16. Solve the equation $\sin\theta\cos\theta = \dfrac{2}{5}$.

$$\sin^2\theta\cos^2\theta = \frac{4}{25};$$

$$\therefore\ \sin^2\theta - \sin^4\theta = \frac{4}{25};$$

$$\therefore\ \sin^4\theta - \sin^2\theta + \frac{1}{4} = \frac{1}{4} - \frac{4}{25} = \frac{9}{100}.$$

$$\therefore\ \sin^2\theta = \frac{1}{2} \pm \frac{3}{10} = \frac{5 \pm 3}{10} = \frac{1}{5} \text{ or } \frac{4}{5}.$$

$$\therefore\ \sin\theta = \pm\frac{1}{\sqrt{5}} \text{ or } \pm\frac{2}{\sqrt{5}};$$

$$\therefore\ \theta = \sin^{-1}\left(\pm\frac{1}{\sqrt{5}}\right) \text{ or } \sin^{-1}\left(\pm\frac{2}{\sqrt{5}}\right).$$

Solve the following equations:

17. $2\sin\theta = \tan\theta$. *Ans.* $\theta = 0$, or 60°.

18. $2\sin^2\theta + 4\cos^2\theta = 3$. *Ans.* $\theta = \pm 45^\circ$.

19. $\tan\theta + 3\cot\theta = 4$. *Ans.* $\theta = 45^\circ$, or $\tan^{-1}3$.

20. $8\sin\theta = 3\cos^2\theta$. *Ans.* $\theta = \sin^{-1}\frac{1}{3}$.

21. $3\sin\theta = 2\cos^2\theta$. *Ans.* $\theta = 30^\circ$.

22. $6\cot^2\theta - 4\cos^2\theta = 1$. *Ans.* $\theta = \pm 60^\circ$.

23. Find the values of $\cos 585^\circ$, $\tan 600^\circ$, $\operatorname{cosec} 690^\circ$.

24. Shew that $\sin^{-1}\frac{4}{5} = \cos^{-1}\frac{3}{5} = \tan^{-1}\frac{4}{3} = \sec^{-1}\frac{5}{3}$.

25. If $\sin(x+\alpha) = \cos(x-\alpha)$, find x. *Ans.* $x = 45^\circ$.

26. If $\sin(\alpha - x) = \cos(\alpha + x)$, find x. *Ans.* $x = -45^\circ$.

EXAMPLES B.

1. If $\operatorname{cosec}\alpha = 1\cdot\dot{3}$, find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\cot\alpha$, $\sec\alpha$.

Prove the truth of the following equations,

2. $$(\sin^2\alpha - \sin^2\alpha\sin^2\beta)(\cos^2\alpha - \cos^2\alpha\cos^2\beta) \\ = (\sin^2\beta - \sin^2\alpha\sin^2\beta)(\cos^2\beta - \cos^2\alpha\cos^2\beta).$$

3. $$\sec^2\alpha\tan^2\alpha(\sec^2\beta - 1)(\tan^2\beta + 1) = \sec^2\beta\tan^2\beta \\ (\sec^2\alpha - 1)(\tan^2\alpha + 1).$$

4. $\sin\alpha(1 + \tan\alpha) + \cos\alpha(1 + \cot\alpha) = \sec\alpha + \operatorname{cosec}\alpha$.

5. $\sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{(1-x^2)}}$.

6. $\sin^{-1}\frac{p-q}{p+q} = \tan^{-1}\frac{p-q}{2\sqrt{pq}}$.

7. $2\operatorname{vers}\frac{180^\circ + \alpha}{2}\operatorname{vers}\frac{180^\circ - \alpha}{2} = \operatorname{vers}(180^\circ - \alpha)$.

8. $\sin\left(\frac{4n+3}{2}180^\circ + \alpha\right) = -\cos\alpha$.

9. $\cos\{(4n+2)\,90^\circ \pm \alpha\} = -\cos\alpha$.

10. If $\sec\theta\tan\theta = 2\sqrt{3}$, then $\theta = 60^\circ$.

11. If $\tan\theta\tan\phi = 1$, then $\sec\theta = \operatorname{cosec}\phi$.

12. If $\frac{\sin\alpha}{\sin\beta} = \sqrt{2}$, $\frac{\tan\alpha}{\tan\beta} = \sqrt{3}$, then $\alpha = 45^\circ$, $\beta = 30^\circ$.

13. If $\dfrac{\cos\theta}{1+\sin\theta}+\tan\theta=2$, $\theta=60^\circ$.

14. If $\cos\phi=n\sin\alpha$, $\cot\phi=\dfrac{\sin\alpha}{\tan\beta}$,

then
$$\cos\beta=\frac{n}{\sqrt{(1+n^2\cos^2\alpha)}}.$$

15. Construct the angle whose secant is $(\sqrt{5}-1)$.

16. Given a the difference of the lengths of the shadows of a vertical stick when the sun's altitude is α, β, respectively; find the length of the stick.

If the length of the stick is a mean proportional between the lengths of the shadows, shew that α, β are complementary.

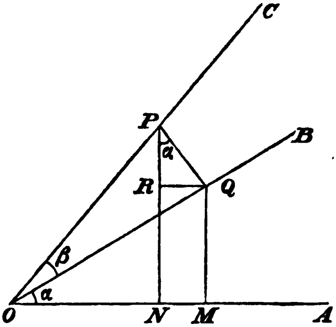
17. The elevation of Cader Idris at a point in the valley near Dolgelly is $\cot^{-1}6$; at Ty Gwyn, $3\frac{1}{2}$ miles down the valley, it has the same elevation; at a point half-way between its elevation is $\cot^{-1}5$. Its height above the valley is $\dfrac{7}{4\sqrt{11}}$ miles.

SECTION III.

TRIGONOMETRICAL RATIOS OF THE SUMS AND DIFFERENCES OF ANGLES, AND OF MULTIPLE ANGLES. EXPLANATION OF THE DOUBLE SIGN IN TRIGONOMETRICAL FORMULÆ.

27. WE now proceed to determine the Trigonometrical Ratios of the sum and difference of any number of angles, in terms of the ratios of the simple angles.

To find the ratios of $(\alpha+\beta)$ in terms of those of α and β.



Let AOB be any angle α, BOC any angle β, then AOC is $(\alpha+\beta)$. In OC take any point P, and draw PN, PQ perpendicular to OA, OB respectively. From Q draw QM, QR perpendicular to OA, PN: then, since RPQ, RQO are each complementary to PQR, therefore $RPQ = RQO = QOM = \alpha$.

And we have,

$$\sin(\alpha+\beta) = \frac{PN}{OP} = \frac{QM}{OP} + \frac{PR}{OP}$$

$$= \frac{QM}{OQ} \cdot \frac{OQ}{OP} + \frac{PR}{PQ} \cdot \frac{PQ}{OP}$$

Now $$\frac{QM}{OQ} = \sin\alpha,\quad \frac{OQ}{OP} = \cos\beta,$$

$$\frac{PR}{PQ} = \cos\alpha,\quad \frac{PQ}{OP} = \sin\beta,$$

therefore $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$;

also $$\cos(\alpha+\beta) = \frac{ON}{OP} = \frac{OM}{OP} - \frac{QR}{OP}$$

$$= \frac{OM}{OQ}\cdot\frac{OQ}{OP} - \frac{QR}{PQ}\cdot\frac{PQ}{OP}$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta;$$

also $$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\beta\sin\alpha}{\cos\alpha\cos\beta - \sin\alpha\sin\beta};$$

divide numerator and denominator of this fraction by $\cos\alpha\cos\beta$, and we have,

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta};$$

this might have been found directly by geometry as in the case of the sine and cosine.

The sine, cosine, and tangent being found as above, we may find the secant, cosecant, and cotangent.

Put $\beta+\gamma$ for β in the above form, and we have,

$$\sin(\alpha+\beta+\gamma) = \sin\alpha\cos(\beta+\gamma) + \cos\alpha\sin(\beta+\gamma)$$

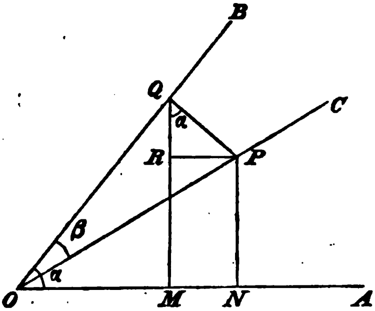
$$= \sin\alpha(\cos\beta\cos\gamma - \sin\beta\sin\gamma) + \cos\alpha(\sin\beta\cos\gamma + \cos\beta\sin\gamma)$$

$$= \sin\alpha\cos\beta\cos\gamma + \sin\beta\cos\alpha\cos\gamma + \sin\gamma\cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\gamma.$$

Similarly the forms for $\cos(\alpha+\beta+\gamma)$, $\tan(\alpha+\beta+\gamma)$ may be found; and the method may be extended to the sum of any number of angles.

28. To find the Trigonometrical ratios of $(\alpha-\beta)$, in terms of those of α and β.

Let AOB be any angle α, BOC any angle β, then AOC is $(\alpha-\beta)$. In OC take any point P, and draw PN, PQ perpendicular to OA, OB respectively; from Q draw QM perpendicular to OA, and from P, PR perpendicular to QM: then $PQR = QOA = \alpha$, for PQR, QOA are each complementary to OQM.



And we have,

$$\sin(\alpha-\beta) = \frac{PN}{OP} = \frac{QM}{OP} - \frac{QR}{OP}$$

$$= \frac{QM}{OQ}\cdot\frac{OQ}{OP} - \frac{QR}{PQ}\cdot\frac{PQ}{OP}$$

$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta;$$

also, $$\cos(\alpha-\beta) = \frac{ON}{OP} = \frac{OM}{OP} + \frac{PR}{OP}$$

$$= \frac{OM}{OQ}\cdot\frac{OQ}{OP} + \frac{PR}{PQ}\cdot\frac{PQ}{OP}$$

$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta;$$

also $$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta};$$

and by means of these formulæ the Trigonometrical ratios of the sums and differences of any number of angles may be calculated.

29. The figures in the above proofs have been drawn so that all the angles are less than 90°. It is left as practice to the student to adapt the proof to any case whatever, and to satisfy himself that the forms proved are true for any angles, a logical proof of which he will find in more advanced treatises on the subject.

Assuming the form for sin $(\alpha+\beta)$ to be true for any values of α and β, the forms for the other ratios might be deduced as follows:

$$\cos(\alpha+\beta) = \sin\{90-(\alpha+\beta)\} = \sin(90+\alpha+\beta) \text{ (Art. 15)}$$
$$= \sin(90+\alpha)\cos\beta + \cos(90+\alpha)\sin\beta$$
$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \text{ as before:}$$

putting $-\beta$ for β,

$$\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta;$$

also
$$\cos(\alpha-\beta) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$$
$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Since $\sin(-\beta) = -\sin\beta$, $\cos(-\beta) = \cos\beta$. (Art. 16).

30. In the forms for $\alpha+\beta$, let $\alpha=\beta$,

then we have,
$$\sin 2\alpha = 2\sin\alpha\cos\alpha,$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha,$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha};$$

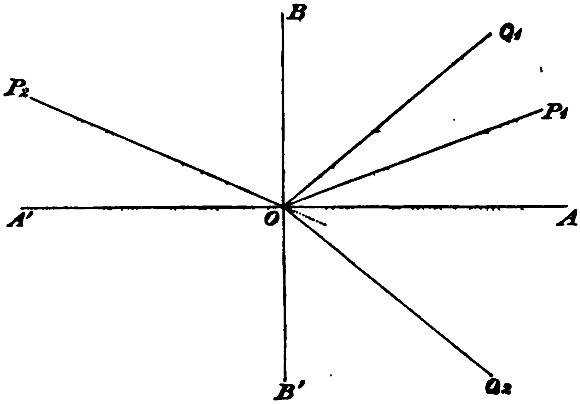
which give the ratios of the double angles in terms of those of the single angles.

If we wish to determine sin 2α, cos 2α in terms of sin α, or of cos α only, we have

$$\sin 2\alpha = \pm 2\sin\alpha\sqrt{(1-\sin^2\alpha)},$$
$$\text{or} = \pm 2\cos\alpha\sqrt{(1-\cos^2\alpha)},$$
$$\cos 2\alpha = 1 - 2\sin^2\alpha,$$
$$\text{or} = 2\cos^2\alpha - 1.$$

31. The double sign in the form for sin 2α may be explained as follows:

Let AOP_1 be the smallest angle whose sine is equal to



sin α: then we cannot tell whether the sin α, from which we are to determine sin 2α, corresponds to the bounding line OP_1 or OP_2, and therefore we ought not, if our formulæ have their proper generality, to be able to tell whether sin 2α corresponds to AOQ_1 or AOQ_2 (i.e. $180^\circ + A'OQ_2$).

Now since $AOQ_2 = 2\,(180^\circ - AOP_1) = 360^\circ - AOQ_1$, therefore the sine of AOQ_2 must be equal to the sine of AOQ_1, and be of opposite sign, in accordance with the formulæ proved above. In the same way, if we consider the negative angles bounded by OP_1, and OP_2 we shall find that our formulæ must correspond to the bounding lines OQ_1, OQ_2, and be correct. So also if we suppose the revolving line to have completed any number of revolutions before reaching the positions OP_1, OP_2.

Precisely similar remarks will apply to sin 2α when determined in terms of cos α.

The student is also recommended to examine why cos 2α, tan 2α have not the double sign.

32. Putting $\beta = 2\alpha$ in the forms for $(\alpha + \beta)$, we have

$$\begin{aligned}\sin 3\alpha &= \sin\alpha\cos 2\alpha + \cos\alpha\sin 2\alpha \\ &= \sin\alpha(1 - 2\sin^2\alpha) + 2\cos^2\alpha\sin\alpha \\ &= \sin\alpha - 2\sin^3\alpha + 2(1 - \sin^2\alpha)\sin\alpha \\ &= 3\sin\alpha - 4\sin^3\alpha;\end{aligned}$$

so also $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$.

And similarly we may find the ratios of any multiple angle.

33. If in the forms for $\sin 2\alpha$ we write $\frac{\alpha}{2}$ for α,

we have, $$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2},$$

also $$1 = \cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}.$$

Hence, $$\pm\sqrt{(1 + \sin\alpha)} = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}.$$

$$\pm\sqrt{(1 - \sin\alpha)} = \cos\frac{\alpha}{2} - \sin\frac{\alpha}{2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}.$$

These formulæ are very useful; the following, which are deduced from them, not so much so.

Adding and subtracting (A) and (B), we have

$$\cos\frac{\alpha}{2} = \pm\sqrt{(1 + \sin\alpha)} \pm \sqrt{(1 - \sin\alpha)},$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{(1 + \sin\alpha)} \mp \sqrt{(1 - \sin\alpha)}.$$

Each of these expressions is capable of four values, two and two of opposite signs, according to the signs of the radicals: the reason of which may be explained in a manner similar to that in Art. 31. In choosing practically which of the signs must be used in any particular case, we must return to the formulæ (A) and (B), and inquire, what the signs of $\cos\frac{\alpha}{2}$, $\sin\frac{\alpha}{2}$ must be, and which is the greatest: we can then determine whether $\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}$ is positive or negative, and so also, whether $\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}$ is positive or negative. Having thus given their proper signs to the radicals, we can add and subtract the equations (A) and (B), and obtain the proper values of $\sin\frac{\alpha}{2}$, $\cos\frac{\alpha}{2}$.

34. Writing $\frac{\alpha}{2}$ for α in the forms for $\cos 2\alpha$, we obtain

$$\cos\alpha=\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2};$$

also
$$\cos\alpha=2\cos^2\frac{\alpha}{2}-1,$$

and therefore
$$\cos\frac{\alpha}{2}=\pm\sqrt{\left(\frac{1+\cos\alpha}{2}\right)},$$

and
$$\cos\alpha=1-2\sin^2\frac{\alpha}{2},$$

and therefore
$$\sin\frac{\alpha}{2}=\pm\sqrt{\left(\frac{1-\cos\alpha}{2}\right)},$$

which give $\cos\frac{\alpha}{2}$, $\sin\frac{\alpha}{2}$ in terms of $\cos\alpha$.

We may also find $\tan\frac{\alpha}{2}$ in terms of $\tan\alpha$, but the formula which results is of very little use.

35. Returning to the formulæ in Arts. 27, 28,

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta,$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta,$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta,$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.$$

We get by addition and subtraction

$$2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta),$$

$$2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta),$$

$$2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta),$$

$$2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta),$$

which forms enable us when we have the product of two sines or cosines to replace them by the sum or difference of two other sines and cosines.

Again, in these forms writing $\frac{\alpha+\beta}{2}$ for α, $\frac{\alpha-\beta}{2}$ for β, and transposing in each equation, we have

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2},$$

$$\sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2},$$

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2},$$

$$\cos\beta-\cos\alpha=2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2},$$

forms which enable us to pass from the sum or difference of two sines or cosines, to the product of two other sines or cosines.

EXAMPLES. A.

1. Find the sine of 15°.

$$\sin 15^\circ=\sin(45^\circ-30^\circ)=\sin 45^\circ\cos 30^\circ-\cos 45^\circ\sin 30^\circ$$

$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Ans. $\sin 15^\circ=\frac{\sqrt{3}-1}{2\sqrt{2}}.$

2. Find the Trigonometrical Ratios of $7\frac{1}{2}°$, $105°$.

3. Prove the formulæ,

$$\sin(\alpha+\beta)\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta.$$

We have $\sin(\alpha+\beta)\sin(\alpha-\beta)$

$$=(\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$$
$$=\sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta$$
$$=\sin^2\alpha(1-\sin^2\beta)-(1-\sin^2\alpha)\sin^2\beta$$
$$=\sin^2\alpha-\sin^2\beta.$$ Q. E. D.

4. $\tan(45°\pm\theta)=\dfrac{1\pm\tan\theta}{1\mp\tan\theta}$.

$$\tan(45°\pm\theta)=\frac{\tan 45°\pm\tan\theta}{1\mp\tan 45°.\tan\theta}=\frac{1\pm\tan\theta}{1\mp\tan\theta}.$$ Q. E. D.

5. If $\sin 2\theta=\cos 3\theta$, find θ and $\sin\theta$.

We have $2\theta+3\theta=90°$, or $\theta=18°$.

Also since $\sin 2\theta=\cos 3\theta$;

$$\therefore\ 2\sin\theta\cos\theta=4\cos^3\theta-3\cos\theta;$$
$$\therefore\ 2\sin\theta=4\cos^2\theta-3$$
$$=1-4\sin^2\theta;$$
$$\therefore\ \sin^2\theta+\frac{\sin\theta}{2}+\frac{1}{16}=\frac{1}{4}+\frac{1}{16}=\frac{5}{16};$$
$$\therefore\ \sin\theta=\frac{-1\pm\sqrt{5}}{4}.$$

Hence $\sin 18°=\dfrac{\sqrt{5}-1}{4}$, since it must be positive.

The other value of $\sin\theta$ corresponds to

$$2\theta+3\theta=-270°,\ \text{or}\ \theta=-54°.$$

6. If $\alpha+\beta+\gamma=180^\circ$,

$$\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma,$$

we have $$\tan(\alpha+\beta)=\tan(180^\circ-\gamma),$$

or $$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=-\tan\gamma,$$

or $$\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma.$$ Q. E. D.

Prove the formulæ.

7. $\cos(\alpha+\beta)\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta.$

8. $\sin(\alpha-\beta)\sin\gamma+\sin(\beta-\gamma)\sin\alpha+\sin(\gamma-\alpha)\sin\beta=0.$

9. $\sin(\alpha+\beta)\cos\alpha-\cos(\alpha+\beta)\sin\alpha=\sin\beta.$

10. $\cos\alpha+\cos(\alpha+2\beta)=2\cos(\alpha+\beta)\cos\beta.$

11. $\cos\beta\cos(2\alpha+\beta)=\cos^2(\alpha+\beta)-\sin^2\alpha.$

12. $\dfrac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{\cos^2\alpha\cos^2\beta}=\tan^2\alpha-\tan^2\beta.$

13. $\sin(45^\circ+\alpha)=\dfrac{1}{\sqrt{2}}(\sin\alpha+\cos\alpha)=\cos(45^\circ-\alpha).$

14. $\sin(60^\circ+\alpha)=\dfrac{1}{2}(\sin\alpha+\sqrt{3}\cos\alpha)=\cos(30^\circ-\alpha).$

15. $\sin(30^\circ+\alpha)+\sin(30^\circ-\alpha)=\cos\alpha.$

16. $\sec(\theta\pm\phi)=\dfrac{\sec\theta\sec\phi}{1\mp\tan\theta\tan\phi}.$

17. $\dfrac{1-\tan(45^\circ-\theta)}{1+\tan(45^\circ-\theta)}=\tan\theta.$

18. $\cos^2(\alpha-\beta)-\sin^2(\alpha+\beta)=\cos 2\alpha\cos 2\beta.$

19. $(\sin\theta-\sin\phi)^2+(\cos\theta-\cos\phi)^2=2\operatorname{vers}(\theta-\phi).$

20. $\cos(30-\alpha)-\cos(30+\alpha)=\sin\alpha.$

21. $\cot\alpha+\tan\alpha=2\operatorname{cosec}2\alpha.$

22. $\cot\alpha - \tan\alpha = 2\cot 2\alpha.$

23. $\dfrac{\sec\alpha - 1}{\sec\alpha + 1} = \tan^2\dfrac{\alpha}{2}.$

24. $\dfrac{\sin\alpha + \sin 3\alpha}{\cos\alpha + \cos 3\alpha} = \tan 2\alpha.$

25. $\tan\alpha + \sec\alpha = \tan\left(45^\circ + \dfrac{\alpha}{2}\right).$

26. $\sec^2\alpha\,\mathrm{cosec}^2\alpha - 2 = \tan^2\alpha + \cot^2\alpha.$

27. $\dfrac{1 + \sin\alpha}{1 + \cos\alpha} = \dfrac{1}{2}\left(1 + \tan\dfrac{\alpha}{2}\right)^2.$

28. $\dfrac{1 + \sin\alpha}{1 - \sin\alpha} = \tan^2\left(45^\circ + \dfrac{\alpha}{2}\right).$

29. $\dfrac{\sin\alpha + \sin 2\alpha}{\cos\alpha - \cos 2\alpha} = \cot\dfrac{\alpha}{2}.$

30. If $\sin x = \sin\alpha\sin(x + y)$, then $\tan x = \dfrac{\sin\alpha\sin y}{1 - \sin\alpha\cos y}.$

31. If $\tan\beta = \dfrac{2\sin\alpha\sin\gamma}{\sin(\alpha + \gamma)}$, then $\cot\alpha$, $\cot\beta$, $\cot\gamma$ are in A. P.

32. If $\sin\beta = m.\sin(2\alpha + \beta)$, then $\tan(\alpha + \beta) = \dfrac{1 + m}{1 - m}\tan\alpha.$

33. If $\cos\theta + \cos 7\theta = \cos 4\theta$, then $\theta = 20^\circ$, or $\dfrac{45^\circ}{2}.$

34. If $\cos\theta - \cos 3\theta = \sin 2\theta$, then $\theta = 0$, or 30°.

35. If $\cos 4\theta + \cos 2\theta = \cos\theta$, then $\theta = 20^\circ$, or 90°.

36. If $\cot\theta = \dfrac{a}{b}$, then $a\sin\theta - b\cos\theta = 0$;

and $\dfrac{a\cos\theta - b\sin\theta}{a\cos\theta + b\sin\theta} = \dfrac{a^2 - b^2}{a^2 + b^2}.$

37. If $\sin x = \dfrac{\sin\alpha}{\sin\gamma}$, $\sin(90 - x) = \dfrac{\sin\beta}{\sin\gamma}$,

then $\cos^2\alpha + \cos^2\beta + \sin^2\gamma = 2$.

Prove the formulæ

38. $\sin^{-1}\dfrac{3}{5} + \cos^{-1}\dfrac{4}{5} = 90°$.

39. $\cot^{-1}a - \cot^{-1}b = \cot^{-1}\dfrac{1+ab}{b-a}$.

40. $\text{vers}^{-1}\dfrac{4a-x}{a} = 180° - \text{vers}^{-1}\dfrac{x-2a}{a}$.

41. $\tan^{-1}\dfrac{3}{5} + \cot^{-1}\dfrac{7}{3} = \cot^{-1}\dfrac{13}{18}$.

EXAMPLES. B.

Prove the formulæ:

1. $\dfrac{\cos\alpha + \sin\alpha}{\cos\alpha - \sin\alpha} = \tan 2\alpha + \sec 2\alpha$.

2. $\dfrac{\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta)}{\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta)} = \tan(\alpha+\beta)$.

3. $16\,\text{cosec}^2 4\alpha - 4\sec^2 2\alpha = \sec^2\alpha\,\text{cosec}^2\alpha$.

4. $\dfrac{\sin\alpha + \sin 2\alpha}{1 + \cos\alpha + \cos 2\alpha} = \tan\alpha$.

5. $\dfrac{\cos(\alpha + 45°)}{\cos(\alpha - 45°)} = \sec 2\alpha - \tan 2\alpha$.

6. $4\sin\dfrac{\theta}{3}\sin\dfrac{180-\theta}{3}\sin\dfrac{180+\theta}{3} = \sin\theta$.

7. $\dfrac{\tan\alpha + \sec\alpha}{\cot\alpha + \text{cosec}\,\alpha} = \tan\left(45° + \dfrac{\alpha}{2}\right)\tan\dfrac{\alpha}{2}$.

8. $\frac{\sin(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma} = \tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma.$

9. $\tan 4\alpha = \frac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}.$

10. $\tan(30^\circ+\alpha)\tan(30^\circ-\alpha) = \frac{2\cos 2\alpha - 1}{2\cos 2\alpha + 1}.$

11. $\frac{\sin^2 2\alpha - 4\sin^2\alpha}{\sin^2 2\alpha + 4\sin^2\alpha - 4} = \tan^4\alpha.$

12. $\frac{\cos 3\alpha + \sin 3\alpha}{\cos\alpha - \sin\alpha} = 1 + 2\sin 2\alpha.$

13. $\tan\alpha\tan(\alpha-\beta)(\cot\alpha + \tan\beta) = \tan\alpha - \tan\beta.$

14. $\frac{1 + \operatorname{cosec}^2\alpha\tan^2\theta}{1 + \operatorname{cosec}^2\beta\tan^2\theta} = \frac{1 + \cot^2\alpha\sin^2\theta}{1 + \cot^2\beta\sin^2\theta}.$

15. $\sin^2\beta + \sin^2(\alpha-\beta) + 2\sin\beta\sin(\alpha-\beta)\cos\alpha = \sin^2\alpha.$

16. $\frac{\sin^2\alpha - \sin^2\beta}{\sin\alpha\cos\alpha - \sin\beta\cos\beta} = \tan(\alpha+\beta).$

17. $1 + \tan\alpha\tan\frac{\alpha}{2} = \sec\alpha.$

18. $\left(\sin\frac{\alpha+\beta}{2} + \cos\frac{\alpha+\beta}{2}\right)\left(\sin\frac{\alpha-\beta}{2} + \cos\frac{\alpha-\beta}{2}\right)$

$= \sin\alpha + \cos\beta.$

19. $\tan^{-1}\frac{1}{\sqrt{2}} + \sin^{-1}\frac{1}{\sqrt{2}} = \tan^{-1}\frac{\sqrt{2}+1}{\sqrt{2}-1}.$

Solve the equations:

20. $5\tan x = \tan 3x.$ *Ans.* $\tan x = \pm\sqrt{\frac{1}{7}}$, or 0.

21. $4\sin\theta\sin(\theta-\alpha) = 2\cos\alpha - 1.$ *Ans.* $\theta = 30 + \frac{\alpha}{2}.$

22. $3\tan^{-1}t=\tan^{-1}5t.$ *Ans.* $t=0$, or $\pm\dfrac{1}{\sqrt{7}}$.

23. $\cos^{-1}x+\cos^{-1}(1-x)=\cos^{-1}\sqrt{(x-x^2)}.$

Ans. $x=0$, or $\frac{1}{2}$, or 1.

24. $\tan^{-1}\dfrac{1}{x-1}-\tan^{-1}\dfrac{1}{x+1}=\tan^{-1}a.$

Ans. $x=\sqrt{\dfrac{2}{a}}$.

25. $\tan(a+x)\tan(a-x)=\dfrac{1-2\cos 2a}{1+2\cos 2a}.$ *Ans.* $x=30°$.

26. $\cos^{-1}\dfrac{1-x^2}{1+x^2}+\tan^{-1}\dfrac{2x}{1-x^2}=240°.$

Ans. $x=\sqrt{3}$, or $-\dfrac{1}{\sqrt{3}}$.

27. $\{\cot(m+n)a+1\}\cos 2ma=\{\cot(m+n)a-1\}\cos 2na.$

Ans. $a=\dfrac{45°}{m-n}$.

28. $\sec(180-a)\cos(90+a)\cos(180-a)$

$=\tan(90-a)\sin(270+a).$

Ans. $a=\pm 45°$.

29. $\cos na+\cos(n-2)a=\cos a.$ *Ans.* $a=90°$, or $\dfrac{60°}{n-1}$.

30. $\operatorname{cosec}^2\dfrac{\theta}{2}-\sec^2\dfrac{\theta}{2}=2\sqrt{3}\operatorname{cosec}^2\theta.$ *Ans.* $\theta=30°$.

31. $\sin\theta-\cos\theta=2\sqrt{2}\cos\theta\sin\theta.$

Ans. $\theta=15°$, or $75°$, or $135°$.

32. If $\tan\beta+\cot\beta=2\sec 2a$, then $a+\beta=45°$.

33. If $\tan(\theta+a)\tan(\theta-\beta)=\tan^2\theta$,

then $\tan 2\theta=\dfrac{2\sin a\sin\beta}{\sin(a-\beta)}$.

34. If $\tan 2\alpha = 2\tan\beta$, and $\tan\gamma = \tan^3\alpha$,

shew that $\beta - \gamma = \alpha$.

35. Shew that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = 45°$.

36. Find $\tan(\alpha+\beta+\gamma)$ in terms of $\tan\alpha$, $\tan\beta$, $\tan\gamma$; and thence shew that if $\alpha+\beta+\gamma = 180^0$,

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma.$$

37. If $\tan(\alpha+\theta) = n\tan(\alpha-\theta)$, then $\sin 2\theta = \frac{n-1}{n+1}\sin 2\alpha$.

38. If $\sin 2\theta = \tan\theta$, then $\text{chd}\,\theta = \sqrt{2-\sqrt{2}}$.

39. If $\sin\theta + \sin\phi = m$, $\cos\theta + \cos\phi = n$,

$$\text{then } m\sin\theta + n\cos\theta = \frac{m^2+n^2}{2} = m\sin\phi + n\cos\phi.$$

40. If $\tan^3\theta = \frac{b}{a}$, then $\frac{a}{\cos\theta} + \frac{b}{\sin\theta} = \{a^{\frac{2}{3}} + b^{\frac{2}{3}}\}^{\frac{3}{2}}$.

41. Find $\cos(\alpha+\beta+\gamma)$, and if $\alpha+\beta+\gamma = 180^0$, shew that

$\cot\alpha + \cot\beta + \cot\gamma = \cot\alpha\cot\beta\cot\gamma + \text{cosec}\,\alpha\,\text{cosec}\,\beta\,\text{cosec}\,\gamma$.

42. If $\alpha+\beta+\gamma = 180^0$,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma = 1.$$

43. If $\cos^{-1}\frac{x}{a} = 2\sin^{-1}\frac{y}{a}$, then $ax + 2y^2 = a^2$.

44. If $\cot(\alpha-\gamma) - \cot(\alpha-\theta) = \cot(\alpha+\gamma) - \cot(\alpha+\theta)$,

then $\sin(\alpha-\gamma)\sin(\theta-\alpha) = \sin(\alpha+\gamma)\sin(\theta+\alpha)$.

45. If α, β, γ are the angles of a triangle, and

$$\cos\beta = \frac{\sin\alpha}{2\sin\gamma},$$

then the triangle is isosceles.

46. An object 6 feet high, placed on the top of a tower, subtends an angle $\tan^{-1}.015$ at a place whose horizontal distance from the foot of the tower is 100 feet: find the height of the tower.

Ans. 173 feet nearly.

47. A person observes the angle subtended by a tower and its spire to be the same; he knows the height h of the tower, and his distance a from it. The height of the spire is

$$\frac{a^2+h^2}{a^2-h^2}h.$$

48. A person at distance of 20 yards from the nearer of two towers in the same straight line with him, and 10 yards apart, observes them to subtend the same angle. Passing the nearer tower a certain distance, he observes them again subtend the same angle, the complement of the former. The heights of the towers are $6\sqrt{5}$, $4\sqrt{5}$ yards respectively.

49. A person at distance a from a tower, observes the elevation α of the tower, and of the top of the flagstaff $(90^\circ-\alpha)$; the height of flagstaff is $2a \cot 2\alpha$.

If the distance from the tower is unknown, and on receding c feet the elevation of tower is $\frac{\alpha}{2}$; then the height of flagstaff is $c \operatorname{cosec} \alpha \cos 2\alpha$,

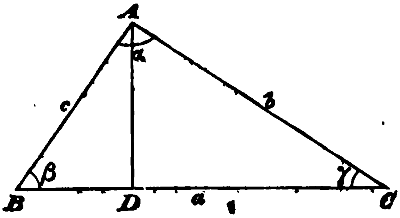
50. From each of two stations in the same horizontal plane at distance D from each other, a pillar on a hill in the same vertical plane with the stations is observed to subtend the same angle α at each of the two stations, and the elevations of the top of the pillar are ϵ, ϵ' at the two stations respectively. The height of pillar is

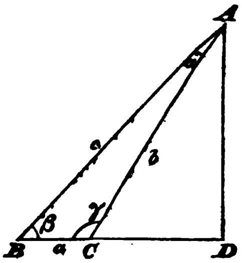
$$\frac{D\sin\alpha}{\sin(\epsilon-\epsilon')}.$$

SECTION IV.

RELATIONS BETWEEN THE SIDES AND ANGLES OF A TRIANGLE. THE RADII OF THE CIRCUMSCRIBED AND INSCRIBED CIRCLES.

36. WE now proceed to establish some important relations between the sides and the Trigonometrical Ratios of the angles of a triangle.





Let ABC be a triangle, from A draw AD perpendicular to BC or BC produced: then denoting the angles of the triangle by α, β, γ, and the sides opposite to them by a, b, c respectively, we have

$$\sin\beta = \frac{AD}{AB}, \text{ or } AD = c\sin\beta;$$

$$\sin\gamma = \sin ACD = \frac{AD}{AC}, \text{ or } AD = b\sin\gamma.$$

Hence $$c\sin\beta = b\sin\gamma,$$

or $$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

Similarly, by dropping a perpendicular from B on AC, we have,

$$\frac{\sin\alpha}{a}=\frac{\sin\gamma}{c};$$

and therefore $$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}.$$

37. Again we have

$$BD = AB\cos\beta \qquad = c\cos\beta,$$

$$CD = AC\cos ACD = b\cos\gamma, \text{ if } \gamma \text{ is acute;}$$

$$= -b\cos\gamma, \text{ if } \gamma \text{ is obtuse.}$$

But $$a = BC = BD + CD \text{ in first case,}$$

$$= BD - CD \text{ in second case;}$$

and therefore in both cases,

$$a = c\cos\beta + b\cos\gamma.$$

This may be derived from the forms in the preceding Article, as follows;

since $\alpha = 180 - (\beta+\gamma)$,

$$c\sin\alpha = c\sin(\beta+\gamma) = c\sin\beta\cos\gamma + c\cos\beta\sin\gamma,$$

or $$a\sin\gamma = b\sin\gamma\cos\gamma + c\cos\beta\sin\gamma;$$

and therefore $$a = b\cos\gamma + c\cos\beta,$$

as before.

Similar forms may be proved for each of the other sides of the triangle.

38. Again $(a - b\cos\gamma)^2 = (c\cos\beta)^2$;

$$\therefore\ a^2 - 2ab\cos\gamma + b^2\cos^2\gamma = c^2 - c^2\sin^2\beta$$

$$= c^2 - b^2\sin^2\gamma;$$

$$\therefore\ c^2 = a^2 + b^2 - 2ab\cos\gamma,$$

and so for the other sides. This may also be put in the form

$$\cos\gamma=\frac{a^2+b^2-c^2}{2ab}.$$

This formula may be obtained geometrically as follows.

By Euclid, II. 13, if ACB is acute,

$$AB^2=BC^2+AC^2-2BC.CD;$$

and by Euclid, II. 12, if ACB is obtuse,

$$AB^2=BC^2+AC^2+2BC.CD.$$

In first case $CD=b\cos\gamma$, in second case $CD=-b\cos\gamma$. Hence in both cases

$$c^2=a^2+b^2-2ab\cos\gamma.$$

39. From the forms in the last Article, we have,

$$1-\cos\gamma=1-\frac{a^2+b^2-c^2}{2ab}=\frac{c^2-(a-b)^2}{2ab},$$

or $$2\sin^2\frac{\gamma}{2}=\frac{(c-a+b)\,(c+a-b)}{2ab}.$$

Let $2s=a+b+c$,

and therefore $s-a=\dfrac{b+c-a}{2}$, and so for $s-b$, $s-c$.

And we have $$\sin^2\frac{\gamma}{2}=\frac{(s-a)\,(s-b)}{ab}.$$

Again, $$1+\cos\gamma=\frac{(a+b)^2-c^2}{2ab}=\frac{(a+b+c)(a+b-c)}{2ab};$$

$$\therefore\ \cos^2\frac{\gamma}{2}=\frac{s\,(s-c)}{ab}.$$

So also $$\tan^2\frac{\gamma}{2} = \frac{(s-a)(s-b)}{s(s-c)}.$$

And $$\sin^2\gamma = 4\sin^2\frac{\gamma}{2}\cos^2\frac{\gamma}{2} = 4\frac{s(s-a)(s-b)(s-c)}{a^2b^2}$$

or $$\sin\gamma = \frac{2}{ab}\sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

40. *To find the area of a triangle.* The area of a triangle is equal to the rectangle contained by half the base, and the perpendicular upon the base from the opposite angle, that is,

$$\text{area } ABC = \frac{1}{2}BC \, . \, AD = \frac{1}{2}ab\sin\gamma\,;$$

this gives the area in terms of two sides and the included angle.

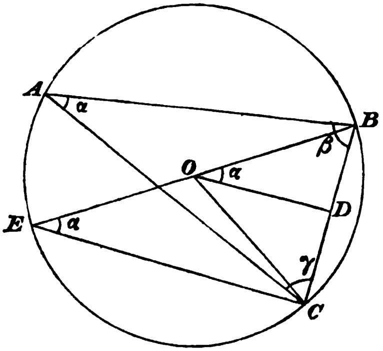
Substituting for $\sin\gamma$ from the preceding article, we have,

$$\text{area } ABC = \sqrt{\{s(s-a)(s-b)(s-c)\}}\,;$$

which is in terms of the sides only.

This expression is frequently denoted by S.

41. *To find the radius of the circumscribed circle.*



Let ABC be a triangle, BE the diameter of the circumscribed circle. Then angle $BEC = BAC = \alpha$; and BCE is a right angle, and we have

$$\frac{BC}{BE} = \sin\alpha, \text{ or } 2R = \frac{a}{\sin\alpha};$$

which gives R in terms of one side and the angle opposite to it, R being the radius of the circle.

Substituting for $\sin\alpha$ in terms of the sides we find,

$$R = \frac{abc}{4S}.$$

These results may also be obtained from the construction for finding the centre of the circumscribed circle given in Euclid, IV. 5.

For BC being bisected in D, and O being the centre, angle BOC = twice the angle at circumference

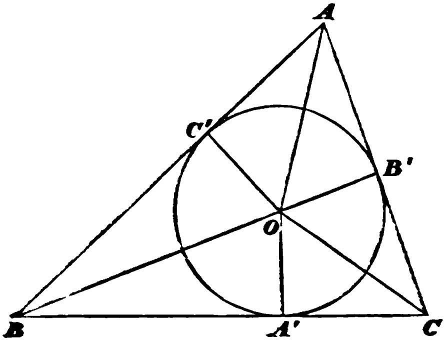
$$= 2\alpha;$$

and therefore $\quad BOD = \alpha, \quad BD = \frac{a}{2},$

and $\quad \frac{a}{2} = R\sin\alpha$, as before.

42. *To find the radius of the inscribed circle.*

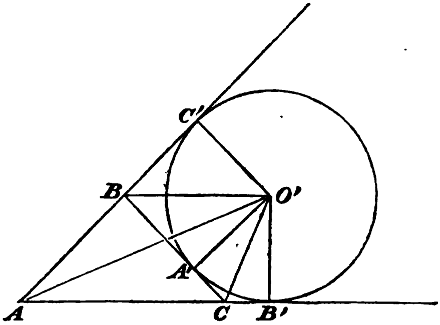
Let O be the centre of the inscribed circle; OA', OB', OC' perpendicular to the sides and equal to r. Then the area of



triangle BOC is $\frac{1}{2}ra$, and so also for triangles BOA, COA. But area of triangle ABC equals the sum of these triangles,

or $$\frac{1}{2}ra+\frac{1}{2}rb+\frac{1}{2}rc=\text{area of } ABC=S;$$

$$\therefore\ r=\frac{S}{s}.$$



To find the radii of the escribed circles, that is, of the circles which touch one side, and the other two sides produced. Let O' be centre of the circle touching BC.

$$O'A'=O'B'=O'C'=r_a.$$

Then $$\Delta ABC=\Delta BAO'+\Delta CAO'-\Delta BCO',$$

or $$S=\frac{1}{2}r_ac+\frac{1}{2}r_ab-\frac{1}{2}r_aa,$$

or $$r_a=\frac{S}{s-a};$$

so also $$r_b=\frac{S}{s-b},\quad r_c=\frac{S}{s-c}.$$

EXAMPLES. A.

Prove the following relations between the sides and angles of a triangle.

1. $a^2 + b^2 + c^2 = 2ab\cos\gamma + 2ac\cos\beta + 2bc\cos\alpha$.

2. $a\,(b\cos\gamma - c\cos\beta) = b^2 - c^2$.

3. $\cos\alpha + \cos\beta = 2\,\dfrac{a+b}{c}\,8\sin^2\dfrac{\gamma}{2}$.

4. $\tan\beta = \dfrac{b\sin\gamma}{a - b\cos\gamma}$.

5. $a^2\sin 2\beta + b^2\sin 2\alpha = 2ab\sin\gamma$.

6. $\cot\alpha - \cot\beta = \dfrac{b^2 - a^2}{ab\sin\gamma}$.

7. $R = \dfrac{1}{8}\,(a+b+c)\sec\dfrac{\alpha}{2}\sec\dfrac{\beta}{2}\sec\dfrac{\gamma}{2}$.

8. $\dfrac{1}{a}\cos^2\dfrac{\alpha}{2} + \dfrac{1}{b}\cos^2\dfrac{\beta}{2} + \dfrac{1}{c}\cos^2\dfrac{\gamma}{2} = \dfrac{(a+b+c)^2}{4abc}$.

9. $1 - \tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2} = \dfrac{2c}{a+b+c}$.

10. $\dfrac{\operatorname{vers}\alpha}{\operatorname{vers}\beta} = \dfrac{a\,(b+c-a)}{b\,(a+c-b)}$.

11. $\dfrac{\cot\dfrac{\beta}{2} + \cot\dfrac{\gamma}{2}}{\cot\dfrac{\alpha}{2}} = \dfrac{2a}{b+c-a}$.

12. $2Rr = \dfrac{abc}{a+b+c}$.

13. $4R\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2} = r$.

14. $r_a r_b + r_a r_c + r_b r_c = s^2$.

15. $a \cos \alpha + b \cos \beta + c \cos \gamma = 4R \sin \alpha \sin \beta \sin \gamma$.

16. $\tan^2 \frac{\alpha}{2} = \frac{rr_a}{r_b r_c}$.

Prove the following expressions for the area of a triangle.

17. $(rr_a r_b r_c)^{\frac{1}{2}}$.

18. $Rr\,(\sin \alpha + \sin \beta + \sin \gamma)$.

19. $\frac{a^2 - b^2}{2} \frac{\sin \alpha \sin \beta}{\sin (\alpha - \beta)}$.

20. $\frac{2abc}{a+b+c} \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$.

21. $\frac{2s^2 \sin \alpha \sin \beta \sin \gamma}{(\sin \alpha + \sin \beta + \sin \gamma)^2}$.

22. The areas of all triangles described about the same circle vary as their perimeters.

EXAMPLES. B.

1. In any triangle $\cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} = \frac{(\text{perimeter})^2}{4 \text{ area}}$.

2. Area of triangle $= s^2 \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2}$.

3. Area of triangle

$$= \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \left(\frac{a^2}{\sin \alpha} + \frac{b^2}{\sin \beta} + \frac{c^2}{\sin \gamma} \right).$$

4. The distance of the foot of the perpendicular from the angle A on the side a, from the middle point of a, is $\frac{b^2 - c^2}{2a}$.

5. If $\cot\alpha$, $\cot\beta$, $\cot\gamma$ are in arithmetical progression, so also are a^2, b^2, c^2.

6. If p_a, p_b, p_c be the perpendiculars from the angles on the opposite sides, then

$$\frac{1}{p_a}+\frac{1}{p_b}+\frac{1}{p_c}=\frac{1}{r}.$$

7. If γ be a right angle,

$$\cos(2\alpha-\beta)=\frac{a}{c^3}(3c^2-4a^2).$$

8. The perpendicular from C on the opposite side AB is

$$\frac{ab}{2}\cdot\frac{a\sin\alpha+b\sin\beta+c\sin\gamma}{ab\cos\gamma+ac\cos\beta+bc\cos\alpha}.$$

9. If $\alpha:\beta:\gamma=2:3:4$, then $\cos\frac{\alpha}{2}=\frac{a+c}{2b}$.

10. The length of the line drawn bisecting the angle α, and meeting the side a is $\frac{2bc}{b+c}\cos\frac{\alpha}{2}$.

11. In any triangle the perpendicular from the angle A on the side BC is equal to $\frac{b^2\sin\gamma\pm c^2\sin\beta}{b\pm c}$.

12. In any triangle

$$\frac{a^2\sin(\beta-\gamma)}{\sin\alpha}+\frac{b^2\sin(\gamma-\alpha)}{\sin\beta}+\frac{c^2\sin(\alpha-\beta)}{\sin\gamma}=0.$$

13. If $A'B'C'$ be the triangle formed by joining the feet of the perpendiculars from the angles A, B, C on the opposite sides, then $B'C'=R\sin 2\alpha$.

14. In any triangle

$$(a+b+c)(\cos\alpha+\cos\beta+\cos\gamma)$$
$$=2a\cos^2\frac{\alpha}{2}+2b\cos^2\frac{\beta}{2}+2c\cos^2\frac{\gamma}{2}.$$

15. If s, s' be the semi-perimeters of two triangles, and $r_a = r'_a$; then $\frac{s-a}{s} : \frac{s'-a'}{s'}$ is the ratio of the radii of the inscribed circles.

16. The angles of a triangle are as 3, 4, 5, and the least side is a; find all the sides.

17. If p, q, r be the perpendiculars from the angles upon the opposite sides a, b, c, respectively; then $\frac{p^2}{qr} = \frac{bc}{a^2}$.

18. The distance of the points of contact of the inscribed circle with the sides AB, AC from the angle A are $s - a$.

19. If a, b, c be in arithmetical progression, then

$$\tan\frac{\alpha}{2}\tan\frac{\gamma}{2} = \frac{1}{3}.$$

20. If the sides of a triangle are in arithmetical progression, then the cotangents of the half angles are in arithmetical progression.

21. In a right-angled triangle (γ the right angle),

$$\sin\frac{\beta}{2} = \sqrt{\left(\frac{c-a}{2c}\right)},$$

SECTION V.

SOLUTION OF TRIANGLES. FOUR CASES. THEIR TREATMENT IN EUCLID. APPLICATION.

43. EVERY triangle has six parts, three angles and three sides; these parts are not independent of each other, for the three angles are always equal to two right angles, that is, we always have

$$\alpha + \beta + \gamma = 180^0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{I.}$$

Also between the sides and angles two other relations independent of each other can be proved to exist. Take for instance the relations proved in Art. 36,

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{II.}$$

$$\frac{\sin \alpha}{a} = \frac{\sin \gamma}{c} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{III.}$$

From I. II. III. all other relations between the sides and angles may be deduced. If we had taken any other two of the relations proved in the last section which are independent of each other, we could with equation I. have deduced all the rest.

Hence since we have three and only three independent equations amongst the six parts of a triangle, if three parts are given, we are enabled to determine the other three. Also three parts at least must be given; but if more were, we should find one or more equations between them to express the condition that these parts may belong to the same triangle.

We may observe also, that the three given parts must not all be angles, for by equation I. if we know two angles we also know the other, and so having three angles given would really only amount to knowing two parts; one part then at least must be a side. Also we must not have two sides given us greater than the third side, nor such as to make the sines and cosines involved in our formulæ greater than unity. (See Art. 46.)

The process of finding the other parts of a triangle, when three of them are known, is called solving the triangle.

44. In solving triangles we shall find four distinct cases present themselves.

CASE I. When the three sides are given.

CASE II. When two sides and an angle opposite to one of them are given.

CASE III. When two sides and the included angle are given.

CASE IV. When two angles and a side are given.

We shall treat each of these in order.

45. CASE I. Let a, b, c be given; to find α, β, γ, we have, Art. 38,

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc},$$

$$\cos\beta = \frac{a^2+c^2-b^2}{2ac},$$

$$\cos\gamma = \frac{a^2+b^2-c^2}{2ab};$$

these equations completely determine the angles α, β, γ, but are not used because they are not adapted for logarithmic computation, as will be shewn hereafter (Art. 61). In practice we use the forms derived from them in Art. 39.

46. CASE II. Let a, b, α be known: to find c, β, γ:

we have $$\sin\beta = \frac{b}{a}\sin\alpha.$$

from which we find β: we then know

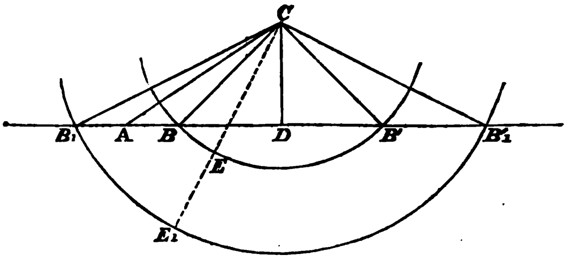
$$\gamma = 180 - (\alpha + \beta),$$

which gives γ: and we then know c from the equation

$$c = a \frac{\sin \gamma}{\sin \alpha}.$$

This case presents a peculiarity which we will now point out.

Let the angle $BAC = \alpha$; $AC = b$; CE, any line drawn from C, equal to a. With centre C and distance CE describe a circle cutting AB in B, B': join CB, CB'; draw CD perpendicular to AB.



If $CE < CD$, the circle will not cut AB at all, and there will be no triangle with the parts proposed, that is, there is no solution.

If $CE > CD$ and $< CA$, two triangles will be formed, CAB, CAB', each having the angle $CAB = \alpha$, the sides $CA = b$ and CB or CB' equal to a: that is to say, there are two solutions of the triangle.

If $CE > CA$, then the circle will meet AB in points B_1, B_1'. Now CAB_1 is not a solution, inasmuch as the angle CAB or α is not a part of it, but CAB_1 or $180 - \alpha$ instead. Hence in this case there is only one solution CAB_1'.

Now let us examine how these cases are indicated by the Trigonometrical formulæ. We have $\sin \beta = \dfrac{b \sin \alpha}{a}$.

Now $CD = b \sin \alpha$, $CE = a$, and if $CE < CD$, $\sin \beta$ is greater than unity, which is impossible, and therefore there can be no solution in this case.

If $CE > CD$ and $< CA$, then $\sin \beta$ is possible, and $b > a$ and therefore $\beta > \alpha$. Now two values β and $180 - \beta$ will satisfy the equation

$$\sin \beta = \frac{b}{a} \sin \alpha,$$

and since β may be either greater or less than 90° both these values will give solutions of the triangle.

If $CE > CA$, $b < a$, and therefore $\beta < \alpha$ and β must be less than 90°; and only the smaller of the values β and $180 - \beta$ will give a solution of the triangle.

From the fact that when $b > a$ there are two solutions of the triangle, this case is called the ambiguous case.

47. Case III. Let a, b, γ be known: to find c, α, β we have

$$c^2 = a^2 + b^2 - 2ab \cos \gamma;$$

this gives the side c: we can then find α and β from the formulæ

$$\sin \alpha = \frac{a}{c} \sin \gamma,$$

$$\sin \beta = \frac{b}{c} \sin \gamma.$$

In determining α, β in this manner we should have to consider the order of magnitude of α, β, γ from knowing that of a, b, c; and so determine whether α or $180 - \alpha$, β or $180 - \beta$ are the true solutions. No ambiguity will really present itself.

The following method will avoid this difficulty as well as be the more useful for logarithms, as will be shewn hereafter.

Since
$$\frac{\sin\alpha}{\sin\beta}=\frac{a}{b},$$

we have
$$\frac{\sin\alpha-\sin\beta}{\sin\alpha+\sin\beta}=\frac{a-b}{a+b},$$

or
$$\frac{2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}}{2\cos\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}}=\frac{a-b}{a+b},$$

or
$$\tan\frac{\alpha-\beta}{2}=\frac{a-b}{a+b}\tan\frac{\alpha+\beta}{2}.$$

Now
$$\frac{\alpha+\beta}{2}=90^\circ-\frac{\gamma}{2};$$

$$\therefore\ \tan\frac{\alpha-\beta}{2}=\frac{a-b}{a+b}\cot\frac{\gamma}{2}.$$

This equation will give $\frac{\alpha-\beta}{2}$, and since $\alpha+\beta$ is known we can find α and β.

The side c will then be found from the equation

$$c=a\frac{\sin\gamma}{\sin\alpha}.$$

48. Case IV. First, let a, β, γ be given; to find b, c, α. We have

$$\alpha=180^\circ-(\beta+\gamma),$$

which gives α, and then b, c are determined from the equations

$$b=a\frac{\sin\beta}{\sin\alpha},$$

$$c=a\frac{\sin\gamma}{\sin\alpha}.$$

Secondly, let a, α, β be given; to find b, c, γ. We have

$$\gamma = 180^\circ - (\alpha + \beta),$$

and then b, c are found as before.

49. It will perhaps be of advantage to the student to compare these cases with the corresponding propositions in Euclid.

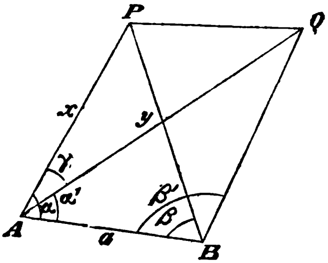
Euclid (I. 8) proves that if two triangles have their sides equal, they have their angles equal; in other words, that if three sides of a triangle are fixed, then the angles cannot vary, which agrees with Case I.

Euclid (I. 4) in the same way agrees with Case III., and (I. 26) with the two parts of Case IV.

Case II. we have already considered geometrically, but the student would do well to compare it with Euclid, VI. 7.

50. We give the following as an example of the application of the rules in this section to practical purposes.

Let it be required to find the distance between the summits P, Q of two inaccessible mountains. At A let the observer measure the angles $PAB = \alpha$, $QAB = \alpha'$, $PAQ = \gamma$; at B let him measure $PBA = \beta$, $QBA = \beta'$, and also let him measure



the distance $AB=a$. Let $AP=x$, $AQ=y$. Then he would proceed with his calculations as follows:

$$AP=x=a\frac{\sin\beta}{\sin(\alpha+\beta)},\quad y=a\frac{\sin\beta'}{\sin(\alpha'+\beta')},$$

which give x and y. PQ is then known from the equation

$$PQ^2=x^2+y^2-2xy\cos\gamma.$$

EXAMPLES. A.

In solving some of these examples, the following values of certain Trigonometrical Ratios will be required.

$\sin 11^\circ.\,32' = \cdot 2 = \cos 78^\circ.\,28'$,

$\sin 16^\circ.\,26' = \cdot 2828 = \cos 73^\circ.\,34'$,

$\sin 21^\circ.\,47' = \cdot 3710 = \cos 68^\circ.\,13'$,

$\sin 32^\circ.\,53' = \cdot 5428 = \cos 57^\circ.\,\ 7'$,

$\sin 38^\circ.\,13' = \cdot 6183 = \cos 51^\circ.\,47'$,

$\sin 45^\circ.\,35' = \cdot 7142 = \cos 44^\circ.\,25'$,

$\sin 68^\circ.\,13' = \cdot 9285 = \cos 21^\circ.\,47'$,

$\tan 10^\circ.\,40' = \cdot 1884 = \cot 79^\circ.\,20'$,

$\tan 50^\circ.\,46' = 1\cdot 2246 = \cot 39^\circ.\,14'$.

1. If $a=3$, $b=7$, $c=5$; solve the triangle.
Ans. $\alpha=21^\circ.\,47'$, $\beta=120^\circ$, $\gamma=38^\circ.\,13'$.

2. If $a=5$, $b=6$, $c=7$; solve the triangle.
Ans. $\alpha=44^\circ.\,25'$, $\beta=57^\circ.\,7'$, $\gamma=78^\circ.\,28'$.

3. If $a=11$, $b=15$, $\gamma=78^\circ.\,28'$; solve the triangle.
Ans. $c=16\cdot 7331$ nearly, $\alpha=40^\circ.\,6'$, $\beta=61^\circ.\,26'$.

4. If $a=3$, $b=5$, $\alpha=21^\circ.\,47'$; find c.
Ans. $c=7$, or $2\cdot 28$ nearly.

5. If $\alpha = 68°.13'$, $\beta = 45°.35'$, $a = 25$; find b.

Ans. $b = 19{\cdot}2$ nearly.

6. If $\alpha = 38°.13'$, $\beta = 21°.47'$, $c = 77$; solve the triangle.

Ans. $a = 55$, $b = 33$, $\gamma = 120°$.

7. Given α, β, and $a + b$; solve the triangle.

8. Given the perimeter $2s$, the area S, and the angle α of a triangle, find the side a.

Ans. $a = s - \frac{S}{s} \cot \frac{\alpha}{2}$.

9. The angles of a triangle are as 3, 4, 5, and the least side is a; find all the sides.

Ans. $\frac{a}{2}\sqrt{6}$, $\frac{a}{2}(\sqrt{3}+1)$.

10. Given the vertical angle, the base, and the difference of the two sides of the triangle; find the other angles.

11. A straight line of length p bisects the angle BAC, and divides the side BC into parts m, n. Shew that

$$p^2 = bc - mn.$$

12. If perpendiculars be drawn from the angles of a triangle upon the opposite sides, and the feet of these perpendiculars be joined to form a new triangle, its sides are $a \cos \alpha$, $b \cos \beta$, $c \cos \gamma$ respectively.

13. Find the height of a cloud by observing its elevation α, and its depression β when seen by reflection in a lake from a station at a height h above the surface of the lake.

Ans. Height above the lake is $h \frac{\sin(\alpha+\beta)}{\sin(\beta-\alpha)}$.

14. AB, CD are two towers in the same horizontal plane. The height of AB is h, the elevation of D at A is α, at B is β. Shew that the height of CD is $\frac{h \sin \alpha \cos \beta}{\sin(\alpha - \beta)}$;

and the distance AC is $\frac{h \cos \alpha \cos \beta}{\sin(\alpha - \beta)}$.

15. Two ships are lying at anchor at a known distance from each other: find by observations made on board the two ships the distance from either of them of an object on the shore.

16. Find c in the equation

$$b^2 + c^2 - 2bc \cos \alpha = a^2,$$

and shew that the ambiguous case is involved in the result.

EXAMPLES. B.

1. In any triangle $\tan\left(\frac{\alpha}{2} + \beta\right) = \frac{c+b}{c-b} \tan \frac{\alpha}{2}$.

2. A circle of radius r is inscribed in a sector of a circle of radius a, the chord of the sector being equal to $2c$; shew that $\frac{1}{r} = \frac{1}{a} + \frac{1}{c}$.

3. Two inaccessible objects C, D are observed from two stations A, B in the same plane with them, so that

$$CAD = \alpha = CBD,\ BAD = \beta,\ ABC = \gamma,\ AB = c:$$

$$\text{shew that } CD = \frac{c \sin \alpha}{\sin(\alpha + \beta + \gamma)}.$$

4. A tower AB stands on an inclined plane; at a point C on the plane the tower is observed to subtend an angle α; and on proceeding to a point D in the line AC, so that $CD = CA$, it subtends an angle β. If ϕ be the angle between the tower and the line AC, then $\cot \phi = \cot \beta - 2 \cot \alpha$.

5. A person in the valley between two hills observes the altitude α of one of them. He then ascends the other through a vertical height a, and finds the angle of depression of his former station β, and the altitude of the hill $(\alpha - \beta)$. The height of the observed hill is $a \frac{\sin^2 \alpha}{\sin^2 \beta}$.

6. A tower situated on a horizontal plane leans towards the north: at two points due south and distant a, b respectively from the foot of the tower, the altitudes of the tower are observed to be α, β respectively: the vertical height of the tower is $\dfrac{b-a}{\cot\beta-\cot\alpha}$, and its inclination to the vertical is

$$\tan^{-1}\frac{b-a}{b\cot\alpha-a\cot\beta}.$$

SECTION VI.

LOGARITHMS. THEIR USE. ADVANTAGE OF LOGARITHMS TO BASE 10. ARRANGEMENT OF THE TABLES. LOGARITHMS OF THE TRIGONOMETRICAL RATIOS.

51. WE have established in the preceding sections certain relations between the sides and angles of triangles by means of which, when we know certain parts of a triangle, we are enabled to calculate the rest: and our examples have illustrated the way in which these relations may be applied to calculate the heights and distances of objects which cannot be directly measured. In most practical cases these methods would involve tedious numerical calculations, and it becomes a matter of great importance to shorten them. This is done by means of what are called logarithms, as we will now proceed to explain.

52. In the equation $N=a^x$, where N, a, x are any quantities satisfying this relation, a is called the *base*, and x the *logarithm* of the number N to the base a. This relation is also thus expressed $x=\log_a N$.

Hence we obtain the following definition. The logarithm of a number to a given base, is the index of the power to which the base must be raised in order to be equal to the given number.

53. In the equation $N=a^x$, let a be some constant quantity greater than unity: then as x gradually increases, N gradually increases;

$$\text{when } x=0,\ N=1;\ \text{ when } x=1,\ N=a;$$

$$\text{when } x=2,\ N=a^2;\ \text{ and when } x=\infty,\ N=\infty.$$

As x increases from 0 to 1, N increases from 1 to a; and whatever value x may have between 0 and 1, there must be some corresponding value of N between 1 and a. And generally, whatever positive value be given to x, there must be some corresponding value of N between 1 and infinity. And if we give x all possible values between 0 and ∞, the corresponding values of N will comprise all possible values between 1 and ∞.

Again, if x receive all possible values from 0 to $-\infty$, N must receive all fractional values between $\frac{1}{a^0}$ and $\frac{1}{a^\infty}$, that is, between 1 and 0.

Hence, by giving x all values from $-\infty$ to $+\infty$, N will have all values from 0 to ∞, that is, all positive values.

And inversely, whatever positive value N may have, there must be some real value of x, such that $a^x = N$.

If a be less than unity, let $a = \frac{1}{b}$, then $a^x = b^{-x} = N$; and by giving x all values from $+\infty$ to $-\infty$, we give b^{-x} all values from 0 to ∞, and therefore a^x or N all values from 0 to ∞.

If $a = 1$, all powers of a are equal to unity, and we cannot make $a^x = N$.

If a is negative, the values of a^x are sometimes positive, sometimes negative, sometimes impossible, and so we cannot always make it equal to N. So also if N is negative we cannot always find x, such that $a^x = N$.

Hence we conclude, that if a is any positive quantity other than unity, we can always find a quantity N equal to a^x whatever real quantity x may be, and we can always find a value of x, such that $a^x = N$, whatever positive quantity N may be.

Now in order to make logarithms of any practical advantage, we must have the logarithms of all numbers up to a certain magnitude, calculated to a given base, arranged in tables; such an arrangement is called a table of logarithms, and the series of numbers and their corresponding logarithms is called a system of logarithms. Hence we see that the base of our system of logarithms must be positive, and the logarithms of negative numbers must be excluded.

We shall assume in the present treatise that the logarithms of all numbers to any positive base other than unity can be calculated to any degree of accuracy, and arranged in tables. The use of such tables is all that falls within our province.

54. Let x, y be the logarithms of the same number N to the bases a, b respectively. Then

$$x = \log_a N, \quad y = \log_b N,$$

and we have

$$a^x = N = b^y;$$

$$\therefore\ a^{\frac{x}{y}} = b;$$

$$\therefore\ \frac{x}{y} = \log_a b;$$

$$\therefore\ \log_b N = \frac{1}{\log_a b} \cdot \log_a N.$$

Hence if we know the logarithm of any number N to base a, and also the logarithm of b to the same base a, we can find the logarithm of N to the base b, by multiplying $\log_a N$ by the fraction $\frac{1}{\log_a b}$. This multiplier is called the modulus of the system whose base is b to the system whose base is a, because by means of it we can reduce a system of logarithms with base a, to a system with base b.

To prove that $\log_a b \times \log_b a = 1$.

Let $\qquad x = \log_a b, \quad y = \log_b a,$

then $\qquad a^x = b, \quad b^y = a,$

$$\therefore\ a^{xy} = (a^x)^y = b^y = a,$$

$$\therefore\ xy = 1;$$

or $\qquad \log_a b \times \log_b a = 1,$

which was to be proved.

55. To find the logarithm of a product or quotient; of a power or root.

Let $x = \log_a M, \quad y = \log_a N,$

then $M = a^x, \quad N = a^y,$

$$\therefore\ MN = a^x . a^y = a^{x+y},$$

$$\therefore\ \log_a (MN) = x + y = \log_a M + \log_a N;$$

similarly $\log_a (MNP\ldots) = \log_a M + \log_a N + \log_a P + \ldots$

Again, $\dfrac{M}{N} = \dfrac{a^x}{a^y} = a^{x-y},$

$$\therefore\ \log_a \left(\frac{M}{N}\right) = x - y = \log_a M - \log_a N.$$

Again, since $M = a^x,$

$$\therefore\ M^r = a^{rx},$$

$$\therefore\ \log_a M^r = rx = r \log_a M.$$

This is true whether r be whole or fractional,

$$\therefore\ \log_a \sqrt[r]{M} = \log_a M^{\frac{1}{r}} = \frac{1}{r} \log_a M.$$

It is on these properties that the great utility of logarithms depends—for the logarithms of the product of two numbers, we here see, is equal to the sum of the logarithms of the numbers, and so a process of multiplication is reduced to one of simple addition. And in the same way we facilitate the operations of division, involution, and evolution, by reducing them to subtraction, multiplication, and division respectively.

Ex. To find the value of $\sqrt[16]{7985683}$.

We find from the tables that

$$\log_{10} 7985683 = 6.9023122,$$

$$\therefore\ \log_{10} \sqrt[16]{7985683} = \frac{6.9023122}{16} = .4313945;$$

But from the tables we have

$$\cdot 4313945 = \log_{10} 2{\cdot}700191,$$

$$\therefore \sqrt[16]{7985683} = 2{\cdot}700191.$$

56. Let the logarithm of any number N to base 10 be $n+d$, where n is some integer, positive, or negative, or zero; and d a decimal fraction; then

$$\log_{10} N = n + d,$$

$$\text{and } \log_{10} 10^m N = \log_{10} 10^m + \log_{10} N = m + n + d.$$

Hence we see that the logarithms of N and $10^m N$ to base 10 have the same decimal part, whatever integer m is, positive or negative. And if we have a system of logarithms calculated to base 10, the same decimal part of the logarithm will correspond to all numbers which differ from each other only in the position of the decimal point, or which can be derived from each other by multiplying or dividing by any power of 10 (numbers of this kind are said to have the same *significant* digits). It is from this property that 10 is chosen as the base of the system of logarithms in common use; and for the future we shall speak of the logarithm of a number to the base 10, as simply the logarithm of the number, and we shall leave out the suffix in writing it, thus by $\log N$ we shall mean $\log_{10} N$.

57. The integral part of a logarithm is called its *characteristic*, the decimal part its *mantissa*. We have shewn that all numbers having the same significant digits have the same mantissæ, we now proceed to shew how the characteristic may be determined in any particular case by simple inspection.

Let N be a number which has one digit before the decimal point, that is to say, some number between 1 and 10: then $\log N$ must lie between 0 and 1, and is therefore some decimal.

Let $$\log N = d,$$

then $10^{n-1} N$ is a number which has n figures before the decimal point, and its logarithm is $(n-1)+d$. Hence we see that if a number has n figures before the decimal point, its characteristic is $(n-1)$.

Again, $\frac{N}{10^{n+1}}$ has n cyphers after the decimal point, before the first significant digit, and its logarithm is $-(n+1)+d$. Hence, if there are n cyphers after the decimal point, the characteristic is $-(n+1)$

This last case requires some explanation:

$$-(n+1)+d=-n-(1-d)=-n-d'=-(n+d'),$$

where $d'=1-d$ and is some decimal. Here the logarithm is negative, and the integral part $-n$: also the mantissa d' of $\log\frac{N}{10^{n+1}}$ is not the same as d the mantissa of $\log N$, which seems to contradict what was before observed: but it is found more convenient to consider all numbers as having logarithms with positive mantissæ, and it is with this convention that we say that $\log N$ and $\log\frac{N}{10^{n+1}}$ have the same mantissæ, and that the characteristic of the latter is $-(n+1)$. Without this convention we should be obliged to have two tables, one for numbers greater, and the other for numbers less than unity.

Ex. $\log 5{\cdot}4952 = {\cdot}7399835$;

$\therefore \log 5495{\cdot}2 = 3{\cdot}7399835$,

and $\log {\cdot}0054952 = \bar{3}{\cdot}7399835$,

the $-$ sign placed above the 3 in the last case signifying that the characteristic alone is negative.

Thus $\bar{3}{\cdot}7399835$ and $-3{\cdot}7399835$ are different: in fact

$$\bar{3}{\cdot}7399835=-3+{\cdot}7399835=-2{\cdot}2600165.$$

58. In the tables in common use the mantissæ of the logarithms of all numbers from 1 to 100000, calculated to seven decimal places, are arranged in order. Thus when we have any number containing only five significant digits we find the mantissa of its logarithm directly from the tables, and prefix the characteristic by inspection; and if we have a logarithm, we can find the logarithm nearest to it in the

tables, and so find the corresponding number correctly to five significant digits.

Numbers of more than five digits must be taken from the tables by means of what are called proportional parts; for an explanation of which the student must consult more advanced treatises on the subject. We will only observe that in determining the side of a triangle of about a mile in length to five digits, the error cannot exceed one inch; a degree of accuracy which is fully equal to that of the ordinary instruments for surveying.

59. Since the Trigonometrical Ratios of angles are real quantities, they also can have logarithms, just in the same way as other quantities. The logarithms of the sines, cosines, tangents, cotangents, secants, cosecants of all angles are arranged in ordinary tables at intervals of one minute. In some tables they are given for every second. They are calculated to seven decimal places as for common numbers; but since they are mostly negative 10 is added to them, to make them positive: hence the tabular logarithms of the Trigonometrical Ratio of any angle is equal to the real logarithm increased by 10. Or as they are written,

$$L \sin \alpha = \log \sin \alpha + 10,$$

and so for the others.

60. Also since $\sin \alpha = \cos (90 - \alpha)$, therefore

$$\log \sin \alpha = \log \cos (90 - \alpha).$$

Hence if we have the log-sines of all angles from 0° to 45°, we also have the log-cosines of all angles from 90° to 45°: and if we have the log-cosines of all angles from 0° to 45°, we have also the log-sines of all angles from 90° to 45°. Hence a complete table of log-sines and log-cosines from 0° to 45° is also a complete table to 90°. This is taken advantage of in the arrangement of the tables. The column headed sines and counted downwards from the top of the page is headed cosines at the bottom of the page and reckoned upwards, as may be best understood by inspecting the tables. The same arrangement applies to the tangents and cotangents, the secants and cosecants.

EXAMPLES. A.

In solving the following examples these logarithms will be required.

$$\log 169408 = 5{\cdot}2289333,$$
$$\log 2 = {\cdot}3010300,$$
$$\log 3 = {\cdot}4771213,$$
$$\log 309017 = 4{\cdot}4899824,$$
$$\log 3727593 = 6{\cdot}5714286,$$
$$\log 4{\cdot}23665 = {\cdot}6270227,$$
$$\log 4{\cdot}45813 = {\cdot}6491525,$$
$$\log 7 = {\cdot}8450980,$$
$$\log 739148 = 5{\cdot}8687314,$$
$$L\sin 18^\circ = 9{\cdot}4899824,$$
$$L\cos 18^\circ = 9{\cdot}9782063.$$

1. Find the logarithms of

309·017, ·0000309017, 309017000.

2. Find the logarithms of all numbers from 1 to 10.

3. Find the logarithms of the Trigonometrical Ratios of 30° and 45°.

4. Find the logarithms of 12, $\sqrt{(45)}$, $\dfrac{\sqrt{2}}{\sqrt[3]{3}}$.

Ans. 1·0791812, ·8266062, $\overline{1}{\cdot}9914746$.

5. Find the value of $\dfrac{\sqrt[3]{3}}{\sqrt[4]{4}\,.\,\sqrt[5]{5}}$. *Ans.* ·739148.

6. Find the value of $\dfrac{3}{7}\left(\dfrac{5}{2}\right)^{\frac{3}{2}}$. *Ans.* 1·69408.

7. Find the value of $(\cdot 0035)^{\frac{1}{7}}$. *Ans.* $\cdot 445813$.

8. If $20^x = 100$, find x. *Ans.* $1\cdot 357$.

9. If $7^x = 2$, find x. *Ans.* $\cdot 356$.

10. Find $L \sin 36^\circ$, $L \tan 18^\circ$.
Ans. $9\cdot 7692187$, $9\cdot 5117760$.

EXAMPLES. B.

1. Find the value of $\left(\frac{1}{3}\right)^{\frac{187}{14}}$. *Ans.* $\cdot 000000423665$.

2. Find the value of $\left(\frac{1}{100}\right)^{\frac{19}{7}}$. *Ans.* $\cdot 000003727593$.

3. Find the value of $\sin 18^\circ$; and shew that the value found above (page 40) is equal to it.

4. If $a^x = \epsilon^z$, find the fraction $\frac{x}{z}$.

5. Shew that $\log_a x : \log_b x = \log_c b : \log_c a$.

6. Transform $\sqrt{(3 \sin \alpha)} = \frac{1}{5}\sqrt{\left(\frac{\cos^3 \beta}{\sin \gamma}\right)}$,

and $$4\sqrt[4]{\left\{\left(\frac{\sin \gamma}{\cos \beta}\right)^3\right\}} = 3\sqrt[3]{\left(\frac{\sin^2 \beta}{\cos^4 \gamma}\right)},$$

into equations between the tabular logarithms of the quantities involved.

7. Assuming that $\log 250$ and $\log 256$ differ by $\cdot 0103$, shew that $\log 2 = \cdot 30103$.

8. Find the logarithm of 9 to base $3\sqrt{3}$, and of 125 to base $\sqrt{5} \,.\, \sqrt[3]{5}$.

Ans. $1\cdot\dot{3}$, $3\cdot 6$.

SECTION VII.

APPLICATION OF LOGARITHMS TO THE SOLUTION OF TRIANGLES; ILLUSTRATION.

61. We have seen that when we know the logarithms of any numbers we can at once find the logarithm of their product and quotient, but that we cannot apply logarithms directly to the determination of the sum or difference of numbers. Hence in calculating any parts of a triangle from the others by logarithms we must use the formulæ which connect the parts by means of factors only, and if we wish to use one of the others we must transform it to the form of a product.

Thus in calculating a from the formulæ

$$\cos a = \frac{b^2 + c^2 - a^2}{2bc},$$

we cannot apply logarithms directly, but if we transform this equation to

$$\cos \frac{a}{2} = \sqrt{\left(\frac{s \,.\, s-a}{bc}\right)},$$

logarithms become immediately applicable.

We may observe by the way, that in using the first equation we should have to perform four operations in multiplication and one in division; in using the latter we should have to look out five logarithms. If a, b, c were simple numbers the first operation would probably give the least trouble, but as in practice they are generally quantities of five or six digits, the latter is much the shortest.

62. In most cases of solution of triangles the formulæ to be employed are in the form of products: the case which offers most difficulty is when two sides are given and the included angle; this we accordingly proceed to discuss.

Let a, b, γ be given; it is required to find the other angles α, β, and the side c.

Since $$\tan\frac{\alpha-\beta}{2}=\frac{a-b}{a+b}\cot\frac{\gamma}{2}. \quad \text{Art. 47,}$$

we have

$$\log\tan\frac{\alpha-\beta}{2}=\log(a-b)-\log(a+b)+\log\cot\frac{\gamma}{2},$$

(supposing $a>b$) which gives $\frac{\alpha-\beta}{2}$; $\frac{\alpha+\beta}{2}$ is already known, and therefore we can easily determine α and β.

Having found α and β we can find c from the equation

$$c=\frac{\sin\gamma}{\sin\alpha},$$

or $$\log c=\log a+\log\sin\gamma-\log\sin\alpha.$$

63. It sometimes happens that in solving this triangle we may know $\log a$, $\log b$, but not a and b, and it becomes an object to save the trouble of looking them out in the tables.

$$\tan\frac{\alpha-\beta}{2}=\frac{a-b}{a+b}\cot\frac{\gamma}{2}$$

$$=\frac{1-\frac{b}{a}}{1+\frac{b}{a}}\cot\frac{\gamma}{2}.$$

Let $\frac{b}{a}=\tan\phi$, an assumption which is always allowable, since the tangent of an angle may have any value;

then $$\tan\frac{\alpha-\beta}{2}=\frac{1-\tan\phi}{1+\tan\phi}\cot\frac{\gamma}{2}$$

$$=\tan(45^\circ-\phi)\cot\frac{\gamma}{2}. \quad \text{See page 40, Ex. 4.}$$

Now $\log \tan \phi = \log b - \log a,$

and is known at once, and therefore ϕ may be found from the tables, and so we know $45^\circ - \phi$, and can then determine $\frac{\alpha - \beta}{2}$ from the equation

$$\log \tan \frac{\alpha - \beta}{2} = \log \tan (45^\circ - \phi) + \log \cot \frac{\gamma}{2}.$$

This method requires four logarithms to be looked out; if we had found a and b first and proceeded by the former method we should have had to look out $\log a$, $\log b$, $\log (a + b)$, $\log (a - b)$, $\log \cot \frac{\gamma}{2}$, $\log \tan \frac{\alpha - \beta}{2}$; in all six logarithms instead of four.

In these formulæ we have used the natural logarithms and not the tabular logarithms of the Trigonometrical Ratios: we must in all cases substitute $L \tan \frac{\alpha - \beta}{2} - 10$ for $\log \tan \frac{\alpha - \beta}{2}$, and similarly for the others.

64. We will illustrate the use of this method by the example in Art. 50.

Let
$$\begin{aligned} \alpha &= 37^\circ.\,30', \\ \alpha' &= 28^\circ.\,40', \\ \beta &= 45^\circ.\,25', \\ \beta' &= 75^\circ.\,36', \\ \gamma &= 16^\circ.\,28', \\ a &= 375{\cdot}98 \text{ yards.} \end{aligned}$$

Then our calculations will be as follows:

$$\begin{aligned} \log x &= \log a + L \sin \beta - L \sin (\alpha + \beta) \\ &= \log 375{\cdot}98 + L \sin 45^\circ.\,25' - L \sin 82^\circ.\,55' \\ &= 2{\cdot}5751647 + 9{\cdot}8526204 - 9{\cdot}9966727 \\ &= 2{\cdot}4311124. \end{aligned}$$

Again, observing that

$$\log\sin(\alpha'+\beta') = \log\sin\{180^\circ-(\alpha'+\beta')\},$$

we have

$$\begin{aligned}\log y &= \log a + L\sin\beta' - L\sin(\alpha'+\beta')\\ &= 2{\cdot}5751647 + 9{\cdot}9861369 - 9{\cdot}9863952\\ &= 2{\cdot}5749064.\end{aligned}$$

Now $$L\tan\frac{P-Q}{2} = L\tan(45^\circ-\phi) + L\cot\frac{\gamma}{2} - 10,$$

where $$\begin{aligned}\log\tan\phi &= \log x - \log y\\ &= -{\cdot}1437940,\end{aligned}$$

and therefore $$\begin{aligned}L\tan\phi &= 9{\cdot}8562060\\ &= L\tan 35^\circ.\,41'.\end{aligned}$$

Hence $$45^\circ - \phi = 9^\circ.\,19',$$

and $$\begin{aligned}L\tan\frac{P-Q}{2} &= 9{\cdot}2149894 + 10{\cdot}8395431 - 10\\ &= 10{\cdot}0545325\\ &= L\tan 48^\circ.\,35'.\end{aligned}$$

Hence $$\begin{aligned}P-Q &= 97^\circ.\,10',\\ P+Q &= 163^\circ.\,32';\\ \therefore\ Q &= 33^\circ.\,11'.\end{aligned}$$

Now $$PQ = \frac{\sin\gamma}{\sin Q}x;$$

$$\begin{aligned}\therefore\ \log PQ &= L\sin\gamma - L\sin Q + \log x\\ &= 9{\cdot}4524879 - 9{\cdot}7382412 + 2{\cdot}4311124\\ &= 2{\cdot}1453591\\ &= \log 139{\cdot}75.\end{aligned}$$

And therefore $PQ = 139{\cdot}75$ yards, the distance required.

EXAMPLES. A.

In addition to the logarithms given before the Examples in the last section, we shall now want the following,

$$\log 1{\cdot}590643 = {\cdot}2015726,$$

$$\log 2{\cdot}07 \quad = {\cdot}3159703,$$

$$\log 3{\cdot}652281 = {\cdot}5625642,$$

$$L\sin 6^\circ.22'.45'' = 9{\cdot}0457574 = L\cos 83^\circ.37'.15'',$$

$$L\sin 23^\circ.42'.43'' = 9{\cdot}6043801 = L\cos 66^\circ.17'.17'',$$

$$L\sin 53^\circ.27'.20'' = 9{\cdot}9049296 = L\cos 36^\circ.32'.40'',$$

$$L\sin 62^\circ.6'.51'' = 9{\cdot}9463950 = L\cos 27^\circ.53'.9'',$$

$$L\tan 10^\circ.53'.36'' = 9{\cdot}2843181 = L\cot 79^\circ.6'.24'',$$

$$L\tan 11^\circ.16'.10'' = 9{\cdot}2994355 = L\cot 78^\circ.43'.50'',$$

$$L\tan 57^\circ.54' = 10{\cdot}2025255 = L\cot 32^\circ.6',$$

$$L\tan 71^\circ.33'.54'' = 10{\cdot}4771213 = L\cot 18^\circ.26'.6''.$$

1. If the sides of a triangle are 32, 40, 66, find the greatest angle. *Ans.* $132^\circ.34'.34''$.

2. The sides a, b, c of a triangle are in the ratio 4, 5, 6, find β. *Ans.* $\beta = 55^\circ.46'.18''$.

3. If $\alpha = 45^\circ$, $a = 14000$, $b = 15906{\cdot}43$, find the other angles. *Ans.* $\beta = 53^\circ.27'.20''$, $\gamma = 81^\circ.32'.40''$,
or $\beta = 126^\circ.32'.40''$, $\gamma = 8^\circ.27'.20''$.

4. If one angle of a triangle is 60°, and the sides containing it are 20, 10; solve the triangle.
Ans. Angles are 30°, 90°, the third side is 17·32.

5. Two sides of a triangle are to each other as 9 : 7, and the included angle is $64^\circ.12'$: find the other angles.
Ans. $69^\circ.10'.10''$, $46^\circ.37'.50''$.

6. If $\alpha = 23^\circ.42'.43''$, $\beta = 18^\circ$, $a = 207$; find b.
Ans. $b = 159{\cdot}0643$.

7. If $\alpha = 60^\circ$, $\beta = 57^\circ.53'.9''$, $c = 3727{\cdot}593$; find a.
Ans. $a = 3652{\cdot}28$.

8. If log a, log b are given, find log $(a+b)$ by means of a subsidiary angle.

Find also log $(a-b)$, where $a > b$.

9. Shew how to find the value of the following expressions by means of subsidiary angles:

$$\sqrt{(a^2 \pm b^2)},\quad a \pm \sqrt{(a^2 - b^2)},\quad \sqrt{(a+b)} \pm \sqrt{(a-b)}.$$

EXAMPLES. B.

1. If two sides of a triangle are 70,35, feet respectively, and the included angle is $36^\circ.52'.12''$; find the remaining angles. *Ans.* $116^\circ.33'.54''$, $26^\circ.33'.54''$.

2. If the angles of a triangle be in arithmetical progression, and the greatest side is to the least in the ratio 5 : 4; find the angles. *Ans.* $70^\circ.53'.36''$, $49^\circ.6'.24''$.

3. One angle of a triangle is 60°, and the ratio of the side opposite to it to the difference of the sides containing it, is $9\sqrt{3} : 2$; find the remaining angles.
Ans. $66^\circ.22'.45''$, $53^\circ.37'.15''$.

4. If $x^2 = a^2 + b^2 + ab$, shew that subsidiary angles may be used to determine x from $x = \sqrt{(ab)} \sec \phi$, where

$$\tan \theta = \frac{b}{a},$$

and
$$\tan^2 \phi = 2 \operatorname{cosec} (2\theta).$$

SECTION VIII.

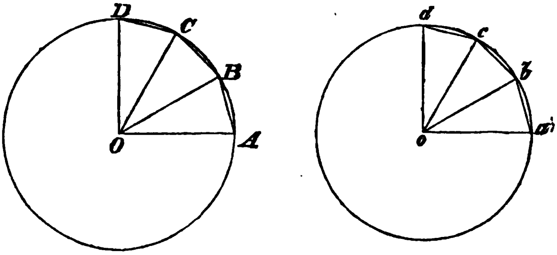
CIRCULAR MEASURE. POLYGONS INSCRIBED, AND CIRCUMSCRIBED ABOUT A CIRCLE. AREA OF A CIRCLE.

65. THERE is another system of measuring angles besides that by degrees or grades, which on account of its importance in all the higher branches of mathematics, it will be necessary to say something about in this treatise.

We must premise the following propositions.

The ratio of the circumference of a circle to its diameter is the same for all circles.

To prove this, we shall assume, what is proved by Newton and appears almost self-evident, that if in a curved figure a polygon be inscribed, and the number of sides of the polygon be increased and the magnitude of each side diminished without limit, the perimeter of the polygon becomes more and more nearly equal to that of the curvilinear figure, and is ultimately equal to it.



Let *ABCD*, *abcd* be two circles whose centres are O, o: and in *ABCD* let a polygon *ABCD...* be inscribed; in *abcd* let a similar polygon *abcd...* be inscribed, then we have

$$AB : AO = ab : ao,$$
$$BC : AO = bc : ao,$$
$$CD : AO = cd : ao,$$
$$\&\text{c}.... \quad = \&\text{c}....$$

Hence $AB + BC + CD + ... : AO = ab + bc + cd + ... : ao$, or Perimeter of polygon $ABCD...$: AO = Per. of pol. $abcd$: ao. Now this proportion is true however many sides the polygons have; let their number be increased, and the magnitudes of each of them diminished indefinitely; then the perimeters of the polygons become the circumferences of the circles, and we have

$$\text{circumference } ABCD : AO = \text{circum. } abcd : ao,$$

or the ratio of the circumference of a circle to its diameter is the same for all circles.

This ratio is generally denoted by the symbol π; so that if r be the radius of a circle, and $2r$ its diameter, its circumference is $2\pi r$. The value of π has been calculated, and is found to be 3·14159 or the fraction $\frac{22}{7}$ nearly.

66. We must next prove that the angle subtended at the centre of any circle by an arc equal to the radius of the circle is of invariable magnitude.

Let AOP (fig. Art. 7) be an angle at the centre of the circle $ABA'B'$, such that the arc AP is equal to the radius AO: then, since angles at the centre of a circle are as the arcs on which they stand (Euclid VI. 33),

$$\frac{\text{angle } AOP}{4 \text{ right angles}} = \frac{\text{arc } AP}{\text{circumference } ABA'B'} = \frac{r}{2\pi r},$$

(since arc AP = radius, and circumference $= 2\pi r$),

$$\text{or} \quad \text{angle } AOP = \frac{4 \text{ right angles}}{2\pi} = \frac{2 \text{ right angles}}{\pi}.$$

Now π being invariable, and this expression independent of r, the angle AOP is the same whatever be the size of the circle.

Hence it appears that this angle is a proper standard by which to measure other angles. If this angle be taken for unity, any other angle which is θ times as great will be represented by θ. θ is called the circular measure of the angle.

67. To shew that θ, the circular measure of an angle, is equal to the arc subtended by that angle, divided by the radius.

Let AOP be the angle whose arc AP = radius AO, AOP_1 any other angle whose circular measure is θ, then

$$\theta = \frac{\text{angle } AOP_1}{\text{angle } AOP} = \frac{\text{arc } AP_1}{\text{arc } AP} = \frac{\text{arc}}{\text{radius}}.$$

Since the arc subtended by four right angles is $2\pi r$, the circular measure of four right angles is $\frac{2\pi r}{r} = 2\pi$. Hence the measure of a right angle is $\frac{\pi}{2}$.

68. To pass from circular measure to degrees and grades and vice versâ, we have, as in Art. 5,

$$\frac{D}{90} = \frac{G}{100} = \frac{2\theta}{\pi}.$$

To find the number of degrees in the unit of circular measure, put $\theta = 1$ in this equation, then we have

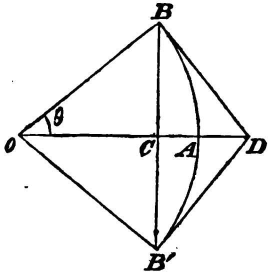
$$D = \frac{180^{\circ}}{\pi} = \frac{180^{\circ}}{3\cdot14159} = 57^{\circ}.29577.$$

69. It is usual to represent angles when degrees are used by the Roman letters A, B, C, ... , when circular measure is used, by the Greek letters $\alpha, \beta, \gamma, \ldots \theta, \phi, \ldots$ We have used the latter throughout this treatise, even when the student had no idea of circular measure. Our reasons for so doing are mentioned in the preface: whilst we are merely dealing with

the relations amongst the Trigonometrical Ratios of various angles, the measure we suppose employed is of no consequence; but when we trace relations between the angles themselves and their Trigonometrical Ratios, we must always use circular measure as in the following article.

70. To shew that $\frac{\sin\theta}{\theta}$, $\frac{\tan\theta}{\theta}$ are equal to unity, when $\theta = 0$.

Let $\angle AOB = \angle AOB' = \theta$, and draw BCB' perpendicular to OA, and let BAB' be the arc of a circle whose centre is O: draw BD, $B'D$ to touch this arc at B, B'.



Then $$\theta = \tfrac{1}{2}\frac{\overset{\frown}{BAB'}}{OA} = \frac{\overset{\frown}{BA}}{OA},$$

$$\tan\theta = \frac{BD}{OB} = \frac{BD}{OA},$$

$$\sin\theta = \frac{BC}{OB} = \frac{BC}{OA}.$$

Hence, since $BD + DB' > \text{arc } BAB'$,

and $BB' < \text{arc } BAB'$,

we have $\tan\theta > \theta$,

and $\sin\theta < \theta$;

Hence $\tan\theta$, θ, $\sin\theta$ are in descending order of magnitude; hence

$$\frac{\sin\theta}{\tan\theta}, \quad \frac{\sin\theta}{\theta}, \quad \frac{\sin\theta}{\sin\theta},$$

or

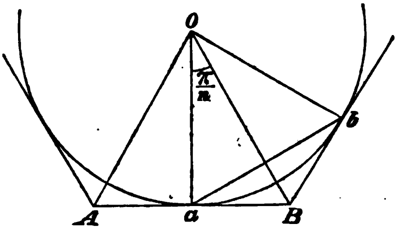
$$\cos\theta, \quad \frac{\sin\theta}{\theta}, \quad 1$$

are in ascending order of magnitude. But when $\theta = 0$, $\cos\theta = 1$, therefore also $\frac{\sin\theta}{\theta}$ must in this case equal unity.

Also

$$\frac{\tan\theta}{\theta} = \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\theta} = 1 \text{ when } \theta = 0.$$

71. To find the perimeters and areas of regular polygons inscribed and circumscribed about a circle.



Let AB, ab be sides of regular polygons of n sides circumscribed and inscribed about a circle whose centre is O.

Then

$$\angle AOB = \angle aOb = \frac{2\pi}{n},$$

$$\therefore \angle aOB = \frac{\pi}{n};$$

Let Oa the radius of the circle $= r$,

Then

$$AB = 2aB = 2r\tan\frac{\pi}{n};$$

perimeter of circumscribed polygon $= n \,.\, AB$

$$= 2nr\tan\frac{\pi}{n};$$

area of circumscribed polygon

$$= n \, . \, aB \, . \, Oa = nr^2 \tan \frac{\pi}{n}.$$

Again, perimeter of the inscribed polygon $= 2nr \sin \frac{\pi}{n}$.

Area of inscribed polygon $= n \, . \, \Delta \, aOb = n \, . \, \frac{1}{2} \, aO \, . \, bO \sin aOb$

$$= \tfrac{1}{2} nr^2 \sin \frac{2\pi}{n}.$$

72. *To find the area of a circle.*

This equals the area of the inscribed polygon when the number of sides is made infinite.

Let $\frac{2\pi}{n} = \theta$, then $n = \frac{2\pi}{\theta}$, and when $n = \infty$, $\theta = 0$.

The area of polygon becomes

$$\frac{\pi}{\theta} r^2 \sin \theta = \pi r^2 \frac{\sin \theta}{\theta},$$

when $$\theta = 0, \quad \frac{\sin \theta}{\theta} = 1;$$

hence, the area of the circle $= \pi r^2$.

EXAMPLES. A.

In the following examples the value of π is supposed to be $\frac{22}{7}$.

1. Find the circular measures of 35°, 35^g.

Ans. $\cdot 6\dot{1}$, $\cdot 55$.

2. Find the degrees &c. in an angle whose circular measure is 1·21. *Ans.* $69^\circ.\ 18'$.

3. The radius of a circle is r; find the length of an arc which subtends A° at the centre. *Ans.* $\frac{\pi r A}{180}$.

4. The radius of the Earth being 4000 miles, what is the length of 1^0 of the meridian? *Ans.* $69\frac{53}{63}$ miles.

5. How far does a person at the equator travel in a second by reason of the Earth's rotation about its axis.
Ans. 256 yards nearly.

6. The diameter of the Earth's orbit about the Sun being 192,000,000 miles, how far in space does the Earth travel in one second? *Ans.* 19 miles nearly.

7. The greatest square possible is cut out of a circle whose radius is r, the area of the remainder is $(\pi - 2) r^2$.

8. The area of a square inscribed in a circle : the area of an equilateral triangle inscribed in the same circle, as $8 : 3\sqrt{3}$.

9. If a_1, a_2, a_3 be respectively the sides of a regular pentagon, hexagon, and decagon inscribed in a circle, then

$$a_1^2 = a_2^2 + a_3^2.$$

EXAMPLES. B.

1. Find the length of an arc subtending an angle of 60^0 in a circle whose radius is 105 feet. *Ans.* 110 feet.

2. If the length of an arc of 60^0 is 11 feet, then the radius of the circle is 10 feet 6 inches.

3. If a right angle were divided into 80 parts, and each of these into 80. Find the number of the latter parts contained in angle whose circular measure is ·001. *Ans.* $4\frac{4}{55}$.

4. Shew that $\frac{\text{vers}\,\theta}{\theta \sin \theta} = \frac{1}{2}$, when $\theta = 0$.

5. Shew that $\frac{\sin m\theta}{\sin n\theta} = \frac{m}{n}$, when $\theta = 0$.

6. A circular flat object at distance a subtends an angle of $30'$, its area is $\frac{\pi^3 a^2}{(720)^2}$ very nearly.

7. If α be the angle whose arc is r_1, and radius r_2, and α' the angle whose arc is r_2 and radius r_1, then

$$\frac{\alpha}{\alpha'} = \left(\frac{r_1}{r_2}\right)^2.$$

8. If R, r be radii of circles circumscribed about, and inscribed in a regular polygon whose side is $2a$, then

$$R^2 - r^2 = a^2.$$

9. The Earth's radius being R and the height of a tower h; find how far an object at sea must be from the foot of the tower so as just to be visible from the top.

Ans. $R \cos^{-1} \frac{R}{R+h}$.

10. The height of one ship is n times that of another, and the top of one is just visible from the masthead of the other at the distance of a miles. Given R the Earth's radius, shew that the height of one is given by the equation

$$\cos^{-1} \frac{R}{R+x} + \cos^{-1} \frac{R}{R+nx} = \frac{a}{R}.$$

11. The radius of the circle of latitude 60° is 2000 miles.

12. What distance is travelled in an hour by a person situated in latitude 60°, by reason of the Earth's rotation?

Ans. $261\frac{19}{21}$ miles.

13. If two plumb-lines suspended from points at a given distance apart on the Earth's surface are inclined at an angle of m''; and at n'' when at an elevation h. Then the radius of the Earth is $\frac{nh}{m-n}$ very nearly.

MISCELLANEOUS EXAMPLES.

1. Prove that the ratios of the angles denoted by a French and English degree, minute, second, are expressed by

$$\frac{3\,.\,3}{2\,.\,5},\ \frac{3\,.\,3^2}{2\,.\,5^2},\ \frac{3\,.\,3^3}{2\,.\,5^3},\ \text{respectively.}$$

2. Shew that $m' = m^{\backprime} + \frac{23}{27}m^{\backprime}$, and $n'' = 3n^{\backprime\backprime} + \frac{7}{81}n^{\backprime\backprime}$.

3. Find the value of the interior angle of any regular polygon of n sides.

Ans. $\frac{n-2}{n}\,180^\circ$.

4. One regular figure has twice as many sides as another, and an angle of the first is one-third as large again as an angle of the second: find the interior angles of each.

Ans. 144°, 108°.

5. The length of the line bisecting the angle a of a triangle and meeting the opposite side is

$$\frac{2\sqrt{bc}}{b+c}\sqrt{\{s\,(s-a)\}}.$$

6. If regular pentagons be inscribed and circumscribed about a circle, then perimeter of the former : perimeter of the latter $= \sqrt{5}+1 : 4$.

7. If $\cos\theta = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos A$,

and $$\sin^2\phi = \sin\alpha\sin\beta\cos^2\frac{A}{2},$$

then $$\sin^2\frac{\theta}{2} = \sin\left(\frac{\alpha+\beta}{2}+\phi\right)\sin\left(\frac{\alpha+\beta}{2}-\phi\right).$$

8. If $\tan x = \cos a \tan y$, then

$$\tan (y - x) = \frac{\tan^2 \frac{a}{2} \sin 2y}{1 + \tan^2 \frac{a}{2} \cos 2y}.$$

9. Eliminate θ between the equations

$$m = \operatorname{cosec} \theta - \sin \theta, \quad n = \sec \theta - \cos \theta.$$

Ans. $(mn)^{\frac{2}{3}} (m^{\frac{2}{3}} + n^{\frac{2}{3}}) = 1.$

10. If $\tan \frac{\alpha}{2} = \tan^3 \frac{\beta}{2}$, and $\tan \beta = 2 \tan \phi$,

then
$$\phi = \frac{\alpha + \beta}{2}.$$

11. If $a \cos \theta + b \cos (\theta + \alpha)$ be put in the form

$$A \cos (\theta + B):$$

find the values of A and B.

Ans. $A = \sqrt{(a^2 + b^2 + 2ab \cos \alpha)}$, $B = \tan^{-1} \dfrac{b \sin \alpha}{a + b \cos \alpha}$.

12. Divide a given angle α into two parts, whose sines are in the ratio $m : n$.

Ans. $\sin^{-1} \dfrac{m \sin \alpha}{\sqrt{(m^2 + n^2 + 2mn \sin \alpha)}}$, $\sin^{-1} \dfrac{n \sin \alpha}{\sqrt{(m^2 + n^2 + 2\,mn \sin \alpha)}}$.

13. If $\cos (\phi - \alpha)$, $\cos \phi$, $\cos (\phi + \alpha)$ are in H. P. then

$$\cos \phi = \sqrt{2} \cos \frac{\alpha}{2}.$$

14. The elevation of a tower standing on a horizontal plane is observed; a feet nearer it is found to be 45°; b feet nearer still it is the complement of what it was at the first station; shew that the height of the tower is $\dfrac{ab}{a - b}$ feet.

15. The distance between the centres of the inscribed circle and of the escribed circle touching the side BC is

$$a \sec \frac{\alpha}{2}.$$

16. Prove the following expression for the area of a triangle,

$$\frac{1}{2}\left(\frac{a^2 \cos\beta \cos\gamma}{\sin\alpha} + \frac{b^2 \cos\alpha \cos\gamma}{\sin\beta} + \frac{c^2 \cos\alpha \cos\beta}{\sin\gamma}\right).$$

17. A number of spheres are put in a hollow cone one above another, and each touching the one above, the one below, and the sides of the cone. Shew that the radii of the spheres form a geometrical progression of which the common ratio is

$$\tan^2\left(45^\circ + \frac{\alpha}{4}\right),$$

α being the vertical angle of the cone.

18. From one of the angles of a rectangle a perpendicular is drawn to its diagonal, and from the point of intersection lines are drawn perpendicular to the sides which contain the opposite angle; shew that if p, p' be the perpendiculars last drawn, and d the diagonal of the rectangle,

$$p^{\frac{2}{3}} + p'^{\frac{2}{3}} = d^{\frac{2}{3}}.$$

19. If two circles whose radii are a, b touch each other externally, and if θ be the angle between their common tangents, shew that

$$\sin\theta = \frac{4(a-b)\sqrt{(ab)}}{(a+b)^2}.$$

20. If $(a+b)\tan(\theta-\phi) = (a-b)\tan(\theta+\phi)$,

$$a\cos 2\phi + b\cos 2\theta = c.$$

Then $$b^2 - a^2 - c^2 + 2ac\cos 2\phi = 0.$$

21. Shew that $\sin 3\theta \sin^3\theta + \cos 3\theta \cos^3\theta = \cos^3 2\theta$.

22. If $\cot\beta - \cot\theta = \cot(\alpha+\theta) - \cot(\alpha-\beta)$,

then
$$\frac{\sin(\alpha-\beta)}{\sin\beta} = \frac{\sin(\alpha+\theta)}{\sin\theta}.$$

23. If $\tan\frac{\alpha}{2} = 2-\sqrt{3}$, find $\sin\alpha$.

24. The elevation of a tower at a place A due south of it is 30^0; and at a place B, due west of A, and at the distance a from it, the elevation is 18^0: shew that the height of the tower is
$$\frac{a}{\sqrt{(2\sqrt{5}+2)}}.$$

25. If $\sin\alpha + \sin(\theta-\alpha) + \sin(2\theta+\alpha)$
$$= \sin(\theta+\alpha) + \sin(2\theta-\alpha),$$

then
$$\theta = \cos^{-1}\frac{\pm\sqrt{5}+1}{4}.$$

26. If $\dfrac{m\tan(\alpha-x)}{\cos^2 x} = \dfrac{n\tan x}{\cos^2(\alpha-x)}$,

then
$$\tan(\alpha-2x) = \frac{n-m}{n+m}\tan\alpha.$$

27. If a, a' be homologous sides of similar triangles inscribed in and described about a given circle, then
$$a = 4a'\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.$$

28. In the ambiguous case in the solution of triangles, if $\alpha, \beta, \gamma, \alpha, \beta', \gamma'$, be the angles of the triangle in the two solutions
$$\frac{\sin\gamma}{\sin\beta} + \frac{\sin\gamma'}{\sin\beta'} = 2\cos\alpha.$$

29. The elevation of two clouds to a person in the same line with them is α. When standing on the shadow of one of them its elevation is 2α, and the other is vertically over him. Shew that the heights of the clouds are as $2\cos^2\alpha : 1$.

30. At each of three stations in the same horizontal plane, and at given distances from each other, the elevation of a tower is observed to be α. Find the height of the tower.

31. In any triangle

$$a\cos 2\beta + 2b\cos\alpha\cos\beta + a\cos 2\gamma + 2c\cos\alpha\cos\gamma = 0.$$

32. If $\sqrt{(a^2 - x^2)}\cos\beta + a\sin\alpha = x\sin\beta$, then

$$x = \mp a\cos(\alpha \pm \beta).$$

33. The elevation of a tower on a horizontal plane is observed: on advancing a feet nearer its elevation is found to be the complement of the former. On again advancing its elevation is found to be double of its first elevation: shew that the last station is $\frac{a}{2}$ feet from the foot of the tower.

34. If the line bisecting the right angle γ of a triangle divide the hypothenuse into parts x, y; then

$$x : y = 1 - t : 1 + t,$$

where $$t = \tan\frac{\alpha - \beta}{2}.$$

35. If $$\sin\theta + \sin 2\theta = m,$$
$$\cos\theta + \cos 2\theta = n.$$

Then $(m^2 + n^2)^2 - 3(m^2 + n^2) - 2n = 0.$

36. $(\sin\alpha + \sec\alpha)^2 + (\cos\alpha + \operatorname{cosec}\alpha)^2 = (1 + \sec\alpha\operatorname{cosec}\alpha)^2.$

37. $\tan^{-1}\{\frac{1}{2}(\cot^2\theta - \tan^2\theta)\} = 2\tan^{-1}(\cos 2\theta).$

38. A person observes that a tower and spire on a hill subtend the same angle β; on ascending the hill a feet, he finds the tower subtends an angle γ and the spire again subtends an angle β: given α the slope of the hill, shew that the height of the spire is

$$\frac{a\sin^2\beta}{\sin(\gamma - \beta)\cos(\alpha + \beta + \gamma)}.$$

39. If the diameter $2R$ of a semicircle be divided into any two parts: and on these parts semicircles be described whose radii are r_1, r_2; then the radius of the circle which touches all three semicircles is

$$\frac{Rr_1r_2}{R^2 - r_1r_2}.$$

40. Shew that $\dfrac{\sin 2\alpha + \sin 4\alpha + \sin 6\alpha}{4 \cos^2 \alpha \cos^2 2\alpha} = \tan \alpha + \tan 2\alpha.$

41. A person on the top of a mountain observes the depression 45° of an object on the plane below him: he then turns through an angle of 30°, and observes the depression of another object on the same plane to be 30°. On descending the mountain he finds the distance between the objects is d. Shew that the height of the mountain is also d.

42. If α, β, γ be in arithmetical progression,

$$\frac{\sin (\alpha + \beta)}{\sin (\alpha - \beta)} + \frac{\sin (\beta + \gamma)}{\sin (\beta - \gamma)} = 2 \sin 2\beta.$$

43. A tower stands in a field in the form of an equilateral triangle, and subtends angles $\tan^{-1} \dfrac{4}{\sqrt{3}}$, $\tan^{-1} \dfrac{4}{\sqrt{7}}$, $\tan^{-1} \dfrac{4}{\sqrt{7}}$ at the three corners: its height is equal to a side of the field.

44. If $$\tan^{-1} \frac{1}{x - 1} - \tan^{-1} \frac{1}{x + 1} = \frac{\pi}{12},$$

then $x = \sqrt{3} + 1$.

45. If $$\tan (\theta + \phi) = m \tan (\theta - \phi),$$

and $$a \cos^2 \phi = b \cos^2 \theta,$$

then $$\cos^2 \theta = \frac{(m + 1)^2 b - (m - 1)^2 a}{(m + 1)^2 b^2 - (m - 1)^2 a^2} a.$$

46. If x, y, z be the lines joining the feet of the perpendiculars from the angles of a triangle upon the opposite sides,

then $\dfrac{x}{a^2} + \dfrac{y}{b^2} + \dfrac{z}{c^2} = \dfrac{a^2 + b^2 + c^2}{2abc}.$

47. If $\cos(\alpha-\beta)\sin(\gamma-\delta)=\cos(\alpha+\beta)\sin(\gamma+\delta)$

then $$\cot\delta=\cot\alpha\cot\beta\cot\gamma.$$

48. Shew that

$$\sin\alpha\cos(\beta-\gamma)-\sin\beta\cos(\gamma-\alpha)=\sin(\alpha-\beta)\cos\gamma.$$

49. Shew that

$$\frac{\cos\alpha+\sin\gamma-\sin\beta}{\cos\beta+\sin\gamma-\sin\alpha}=\frac{1+\tan\frac{\alpha}{2}}{1+\tan\frac{\beta}{2}}\text{ if }\alpha+\beta+\gamma=90^\circ.$$

50. Shew that

$$\frac{2\operatorname{cosec}2\theta-\sec\theta}{2\operatorname{cosec}2\theta+\sec\theta}=\cot^2\left(45^\circ+\frac{\theta}{2}\right).$$

51. If $\alpha+\beta+\gamma=0$,

$$\cos\alpha+\cos\beta+\cos\gamma+1=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

52. If the area of a triangle be constant, the sum of the cotangents of the angles, varies as the sum of the squares of the sides.

53. A column on a pedestal 20 feet high subtends an angle 45° to a person on the ground; on approaching 20 feet it again subtends an angle of 45°. The height of the column is 100 feet.

54. Shew that

$$\cos^2\sigma+\cos^2(\sigma-\theta)+\cos^2(\sigma-\phi)+\cos^2(\sigma-\psi)$$
$$=2+2\cos\theta\cos\phi\cos\psi,$$

$$\text{where }\sigma=\frac{\theta+\phi+\psi}{2}.$$

55. If $\tan\theta = \cos 2\alpha$,

$$\text{then } \sin 2\theta = \frac{1 - \tan^4\alpha}{1 + \tan^4\alpha}.$$

56. Shew that

$$2\tan\left\{\cos^{-1}\frac{1}{\sqrt{(1+x^2)}} - \cos^{-1}\frac{x}{\sqrt{(1+x^2)}}\right\} = x - \frac{1}{x}.$$

57. Shew that in a triangle

$$(r_a - r)(r_b - r)(r_c - r) = 4Rr^2.$$

58. If $a'b'c'$ be the perpendiculars from the centre of the circumscribed circle upon the sides a, b, c respectively, then

$$\frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'} = \frac{1}{4}\cdot\frac{abc}{a'b'c'}.$$

59. Shew that $\cos^{-1}\frac{1}{2} - \sin^{-1}\frac{1}{3} = \tan^{-1}\left(\frac{9\sqrt{3} - 8\sqrt{2}}{5}\right)$.

60. If $\alpha = \tan^{-1}\frac{1}{7}$, $\beta = \tan^{-1}\frac{1}{3}$, then

$$\cos 2\alpha = \sin 4\beta.$$

61. If $2\tan x + \tan(\alpha - x) = \tan(\beta + x)$, then

$$\tan x = \frac{\sin(\alpha - \beta)}{2\sin\alpha\sin\beta}.$$

62. From the top of a tower the depressions α, β of two objects in same horizontal plane with the foot of the tower are observed, and also the angle ω which they subtend; and the distance a between them is known; the height of the tower is

$$\frac{a\sin\alpha\sin\beta}{\sqrt{(\sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\cos\omega)}}.$$

63. Shew that

$$\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)$$
$$=4\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha+\gamma}{2}\sin\frac{\beta+\gamma}{2}.$$

64. If $\sin\theta+\sin\phi=m,$

and $\cos\theta+\cos\phi=n.$

Then $$\cos(\theta+\phi)=\frac{n^2-m^2}{n^2+m^2}.$$

65. Shew that $\tan^{-1}\dfrac{x\cos\phi}{1-x\sin\phi}-\tan^{-1}\dfrac{x-\sin\phi}{\cos\phi}=\phi.$

66. If the tangents of the half angles of a triangle be in arithmetical progression, shew that the cosines of the whole angles are so also.

67. Shew that

$$\cos^{-1}\frac{1-m^2}{1+m^2}-\cos^{-1}\frac{1-n^2}{1+n^2}=\tan^{-1}\frac{2(m-n)(1+mn)}{(1+mn)^2-(m-n)^2}.$$

68. If the straight line which bisects the angle BAC of a triangle, divides the side BC in the ratio $m:n$, then

$$\tan\beta=\frac{n\sin\alpha}{m-n\cos\alpha}.$$

69. Prove that the product of the perpendiculars from the angles on the opposite sides equals

$$\frac{\{(a+b+c)r\}^3}{abc}.$$

70. If lines be drawn from the angles of a triangle to the centre of the inscribed circle, and the points where these lines meet the circle are joined so as to form a new triangle: shew that its sides are in the ratio

$$\sin\frac{\alpha}{4}+\cos\frac{\alpha}{4}:\sin\frac{\beta}{4}+\cos\frac{\beta}{4}:\sin\frac{\gamma}{4}+\cos\frac{\gamma}{4}.$$

71. Eliminate θ and ϕ from the equations

$$a \sin^2 \theta + b \cos^2 \theta = \alpha,$$
$$b \sin^2 \phi + a \cos^2 \phi = \beta,$$
$$a \tan \theta = b \tan \phi.$$

Ans. $\frac{1}{a} + \frac{1}{b} = \frac{1}{\alpha} + \frac{1}{\beta}.$

72. Shew that the value of $a \cos 2\theta + b \sin 2\theta$, when $\tan \theta = \frac{b}{a}$ is a.

73. Find all the values of x which satisfy the equation

$$\operatorname{vers}^{-1}(1+x) - \operatorname{vers}^{-1}(1-x) = \tan^{-1} 2\sqrt{(1-x^2)}.$$

Ans. $\pm 1, \frac{1}{2}.$

74. In the three edges of a cube which meet at one angle, three points A, B, C are taken at distances a, b, c from that angle respectively. The area of the triangle ABC is

$$\frac{1}{2}\sqrt{(a^2b^2 + a^2c^2 + b^2c^2)}.$$

75. Determine x from the equation $\tan(\alpha + x) = m \tan x$, and explain the result when $m = 1$.

76. If AD, AE be drawn bisecting the interior and exterior angles at A of a triangle ABC, and r_1, r_2, r_3, r_4 be the radii of the circles circumscribing the triangles ABD, ACD, ABE, ACE respectively, then $r_1 r_4 = r_2 r_3$.

77. Having given the area, base, and vertical angle of a triangle, find the other angles.

78. Having given two sides of a triangle, and the difference of the angles opposite them, determine the angles.

79. If the angle α be divided into two parts so that their versed sines may be in the ratio $m : n$, one part is

$$2 \cot^{-1} \left\{\cot \frac{\alpha}{2} + \sqrt{\frac{n}{m}} \operatorname{cosec} \frac{\alpha}{2}\right\}.$$

80. If r, r' be radii of the circumscribed and inscribed circles of a regular polygon of n sides, shew that

$$r + r' = a \cot \frac{\pi}{2n},$$

where a is one of the sides.

81. Find the number of acres in a field whose sides are 400, 300, 300, 300 yards respectively, and one angle adjacent to the largest side is a right angle.

Ans. $\frac{125}{242}(5\sqrt{11} + 24)$ acres.

82. In a triangle

$$\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} + \sin^2\frac{\gamma}{2} + 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = 1.$$

83. Shew that $\dfrac{\sin\alpha \pm \sin n\alpha + \sin(2n-1)\alpha}{\cos\alpha \pm \cos n\alpha + \cos(2n-1)\alpha} = \tan n\alpha.$

84. If the sides of a triangle are in arithmetical progression, the perpendicular on the mean side from the opposite angle, and the radius of the escribed circle touching the mean side, are each three times the radius of the inscribed circle.

85. If

$$a\cos^2\theta + b\sin^2\theta = m\cos^2\phi,$$

$$a\sin^2\theta + b\cos^2\theta = n\sin^2\phi,$$

$$m\tan^2\theta = n\tan^2\phi,$$

then

$$(a+b)(m+n) = 2mn.$$

86. If $\alpha + \beta + \gamma = 180^\circ$,

$$\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

87. If $\alpha + \beta + \gamma = 180^\circ$, then

$$\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2} = \cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2}.$$

88. The distance between the centres of two wheels is a, and the sum of their radii is c: shew that the length of a string which wraps round them and crosses between them is

$$2\sqrt{(a^2-c^2)}+c\left(\pi+2\sin^{-1}\frac{c}{a}\right).$$

89. An object a feet high at the top of a tower subtends a given angle α at a distance d from the foot of the tower. The tower's height is

$$-\frac{a}{2}\pm\frac{\sqrt{\{a^2+4d(a\cot\alpha-d)\}}}{2}.$$

Explain the negative result.

90. If $\cos^{-1}x+\cos^{-1}(1-x)=\cos^{-1}(-x)$, then $x=\frac{1}{2}$ or 0.

91. In a triangle, if a, b, c be in arithmetical progression, then

$$\sin\left(\alpha+\frac{\beta}{2}\right)=2\sin\frac{\beta}{2}.$$

92. Shew that $\cot\gamma=\frac{b}{c}\operatorname{cosec}\alpha-\cot\alpha$, and adapt the result to logarithmic computation.

93. Shew that $\tan\theta\tan(60+\theta)\tan(60-\theta)=\tan 3\theta$.

94. If $\dfrac{\sin(x+a)}{\cos(x-a)}=\dfrac{1-m}{1+m}$, then

$$\tan(45^\circ-x)=m\cot(45^\circ-a).$$

95. The hypothenuse AB of a right-angled triangle ACB is divided in D so that $AD : BD = CB : CA$. Shew that

$$\tan ACD=\frac{a^2}{b^2},\quad CD=\frac{\sqrt{(a^4+b^4)}}{a+b}.$$

96. If $\tan^{-1}x+\frac{1}{2}\sec^{-1}5x=45^\circ$, then $x=\pm\frac{1}{3}$,

97. If the lines drawn from the angles of a triangle perpendicular to the opposite sides meet in a point whose distances from the three angles are p, q, r, respectively, then the area of the triangle is

$$\frac{1}{4}(pa+qb+rc).$$

98. The area of the triangle formed by joining the centres of the escribed circles is $\dfrac{abc\sqrt{s}}{2\sqrt{\{(s-a)(s-b)(s-c)\}}}$.

99. Shew that

$$\sec^2(\alpha+45^\circ)-\sec^2(\alpha-45^\circ)=4\tan 2\alpha\sec 2\alpha.$$

100. If $\cos(\sigma-2\alpha)+\cos(\sigma-2\beta)=\cos(\sigma-2\gamma)+\cos(\sigma-2\delta)$, where $\sigma=\alpha+\beta+\gamma+\delta$, then $\tan\alpha\tan\beta=\tan\gamma\tan\delta$.

101. One angle of a triangle is 60°, and the sides including it are in the ratio 5 : 3. The other angles are

$$\tan^{-1}\frac{3\sqrt{3}}{7}\text{ and }\tan^{-1}5\sqrt{3}.$$

102. If the sides of a triangle ABC be divided into parts at D, E, F which have to one another the same ratio $m:1$. Shew that if points D, E, F be joined, the triangles ADF, BDE, CEF are all equal, and that

$$\Delta DEF:\Delta ABC=n^2-n+1:(n+1)^2.$$

103. If $\cos\alpha=\cos\beta\cos\phi=\cos\beta'\cos\phi'$,

$$\text{and }\sin\alpha=2\sin\frac{\phi}{2}\sin\frac{\phi'}{2}.$$

Shew that $\tan\dfrac{\alpha}{2}=\tan\dfrac{\beta}{2}\tan\dfrac{\beta'}{2}$.

104. A person walking along a straight road observes that the greatest angle which two objects make with each other is α: from the point where this happens he walks a yards, and the objects there appear in the same straight line

making an angle β with the road: the distance between the objects is

$$\frac{2a \sin\alpha \sin\beta}{\cos\alpha - \cos\beta}$$

105. Shew that

$$2\tan^{-1}\left[\tan\frac{\alpha}{2}\sqrt{\left\{\tan\left(\frac{\pi}{4}-\beta\right)\right\}}\right] = \cos^{-1}\left(\frac{\cos\alpha + \tan\beta}{1+\cos\alpha\tan\beta}\right).$$

106. A circle is described about a triangle ABC, and a new triangle is formed by joining the points of bisection of the arcs subtended by the sides of ABC. Shew that the angles of this triangle are

$$90-\frac{\alpha}{2},\quad 90-\frac{\beta}{2},\quad 90-\frac{\gamma}{2}.$$

And the sides are

$$\frac{a}{2\sin\frac{\alpha}{2}},\quad \frac{b}{2\sin\frac{\beta}{2}},\quad \frac{c}{2\sin\frac{\gamma}{2}}:$$

and its area : area of triangle $ABC = 1 : 8\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$.

107. Three circles whose radii are a, b, c touch each other externally; prove that the tangents at the points of contact meet in a point whose distance from any one of them is

$$\sqrt{\left(\frac{abc}{a+b+c}\right)}.$$

108. If $a = \operatorname{cosec}\theta \tan^3\theta\,(\operatorname{cosec}^2\theta + 1) + \sec\theta$,

and $b = \tan^3\theta\,(\operatorname{cosec}^2\theta + 1) - \tan\theta$,

then $a^{\frac{2}{3}} - b^{\frac{2}{3}} = 2^{\frac{2}{3}}$.

109. A person wishing to determine the length of an inaccessible wall places himself due south of one end, and due west of the other, at such distances that the angles which the wall subtends at the two positions are each equal to α. If q be

the distance between the two positions, the length of the wall is $a \tan \alpha$.

110. A vessel observed another vessel α^0 from the north sailing in a direction parallel to its own. After an hour's sailing its bearing was β^0, and after another hour γ^0. In what direction were the vessels sailing?

$$\textit{Ans.}\quad \frac{\alpha+\gamma}{2} + \tan^{-1}\left\{\tan^2\frac{\gamma-\alpha}{2}\cot\left(\beta-\frac{\alpha+\gamma}{2}\right)\right\},$$

$$\text{or otherwise,}\quad \tan^{-1}\frac{\tan\beta\,(\tan\alpha+\tan\gamma)-2\tan\alpha\tan\gamma}{2\tan\beta-(\tan\alpha+\tan\gamma)}.$$

111. A circle is inscribed in a triangle and tangents drawn parallel to each side and intercepted by the other two. If l, m, n be the lengths of the tangents opposite the sides a, b, c respectively, then

$$\frac{l}{a}+\frac{m}{b}+\frac{n}{c}=1.$$

112. A circle is inscribed in a triangle; similar triangles are cut off by tangents to the circle, and in these triangles are inscribed other circles. Shew that the radius of the first is equal to the sum of the radii of the others.

113. Two squares may be inscribed in a right-angled triangle, one having two sides lying on the sides of the triangle, and the other having one side on the hypothenuse. Shew that the latter is always the greatest.

114. The distance between the centres of the inscribed and circumscribed circles is $\sqrt{(R^2-2Rr)}$.

115. In the triangle ABC, $B'C'$ is drawn through A parallel to BC, and $A'B'$ through C perpendicular to AC, and $A'C'$ through B perpendicular to AB. Shew that the area of the triangle $A'B'C'$ is

$$\frac{a^2}{2}\cdot\frac{\cos^2(\beta-\gamma)}{\cos\beta\cos\gamma\sin\alpha}.$$

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